Algebraic Geometry Problems

Universal assumption: k is a field. All rings are commutative with 1.

1. Show that
   - If $I \subset J \subset k[x_1, \ldots, x_n]$ then $V(J) \subset V(I)$
   - $V(\sum I_\alpha) = \cup V(I_\alpha)$ for any family of ideals $I_\alpha \subset k[x_1, \ldots, x_n]$.
   - If $X \subset Y \subset \mathbb{A}^n$ then $I(Y) \subset I(X)$.
   - $I(X \cup Y) = I(X) \cap I(Y)$.
2. Show that X is closed and find $I(X)$ if
   - $X = \{(0,1), (1,0)\} \subset \mathbb{A}^2$.
   - $X = \{(t, t^2, t^3) \mid t \in k\} \subset \mathbb{A}^3$
   - X is the union of the x-axis and the yz-plane in $\mathbb{A}^3$.
3. Show that an algebraically closed field is infinite.
4. Find $\sqrt{(y^2 - x^3, x^2 + y)} \subset k[x, y]$
5. The Noether Normalisation theorem is the following: Let A be a finitely generated ring over the field k. Then there exist algebraically independent elements $x_1, \ldots, x_d \in A$ such that A is a finitely generated $k[x_1, \ldots, x_n]$-module.

   Prove the Noether normalisation theorem when k is an infinite field. Here is one strategy

   - Show that we can find generators $x_1, \ldots, x_n$ of A ordered in such a way that $x_1, \ldots, x_d$ are algebraically independent, and $x_{d+1}, \ldots, x_n$ are all algebraic over $k[x_1, \ldots, x_d]$.
   - Show that after making a linear change of variables, we can ensure that each of $x_{d+1}, \ldots, x_n$ is the root of a monic polynomial with coefficients in $k[x_1, \ldots, x_d]$.
   - Deduce Noether normalisation.

   Bonus: Can you work out how to modify the second step when k is finite?
6. Let k be an infinite field and $f(x_1, \ldots, x_n)$ be a nonzero polynomial in n variables. Prove that there exists $a_1, \ldots, a_n \in k$ such that $f(a_1, \ldots, a_n) \neq 0$.
7. Let $R \to S$ be a ring homomorphism. The following are equivalent
   - $R \to S$ is finite,
   - $R \to S$ is integral and of finite type,
   - there exist $x_1, \ldots x_n \in S$ which generate S as an algebra over S such that each x_i is integral over R.

8. Prove the weak Nullstellensatz. This is the statement that if I is a proper ideal of $k[x_1, \ldots, x_n]$ then $V(I) \neq \emptyset$. *Hint, use Noether normalisation.*

9. Prove the Nullstellensatz. *Hint: The hard part is to show that $I(V(J)) \subset \sqrt{J}$. Let $J = (g_1, \ldots, g_m)$ and suppose $f \in I(V(J))$. Move up one dimension and consider the ideal $J' = (g_1, \ldots, g_m, ft-1) \subset k[x_1, \ldots, x_n, t]$. Apply the weak Nullstellensatz to obtain an equation writing 1 as a combination of the g_i and $ft-1$, then make a substitution $t = 1/f$.*

10. Let $X \subset \mathbb{P}^n$ be a closed subvariety, and f a non-constant homogeneous element of the projective coordinate ring $R = k[x_0, x_1, \ldots, x_n]/I(X)$. Prove that $D_+(f) = \{x \in X \mid f(x) \neq 0\}$ is an open affine subvariety of X with coordinate ring $R_{(f)}$ (the degree zero elements of $R[1/f]$).

11. Let $n, d \geq 1$ be integers. Consider the map $\phi_{n,d} \colon \mathbb{P}^n \longrightarrow \mathbb{P}^N$, called the Veronese embedding, given by

$$\phi_{n,d}([x_0 : x_1 : \cdots : x_n]) = [\underline{x}^{\underline{i}}]_{|\underline{i}|=d}.$$

Show that

- $N = \binom{d+n}{n} - 1$.
- The image is equal to the zero locus of the ideal generated by the quadratic equations $z_{\underline{i}} z_{\underline{j}} = z_{\underline{k}} z_{\underline{l}}$ where $\underline{i} + \underline{j} = \underline{k} + \underline{l}$.
- $\phi_{n,d}$ is a homeomorphism onto its image.

12. The Cayley-Hamilton theorem says the following: Let A be a $n \times n$ matrix with entries in k. Let $P_A(t) = \det(tI - A)$ be its characteristic polynomial. Then $P_A(A) = 0$. Prove the Cayley-Hamilton theorem in the following way: Show that the xet of diagonalisable matrices contains a Zariski open subset $U \subset \mathrm{Mat}(n, k)$ of matrices with n distinct eigenvalues. Consider the regular function $P_A(A)$ on $\mathrm{Mat}(n, k)$ and show that it is identically zero on U. Conclude that it is identically zero on $\mathrm{Mat}(n, k)$.

13. Suppose the characteristic of k is not three. Find all singular points on the Fermat surface $X_0^3 + X_1^3 + X_2^3 + X_3^3$ in $\mathbb{P}^3$.

14. Consider the plane curve $y^2 = x^3 + Ax + B$. Find conditions on A and B such that this curve is nonsingular. Show that it has a unique point at infinity in $\mathbb{P}^2$ and check that this is always a smooth point.

15. Let $n \geq 4$ be an integer and $a_1, \ldots, a_n \in k$. Show that the plane curve $y^2 = (x - a_1)(x - a_2) \cdots (x - a_n)$ has a unique point at infinity and that it is not a smooth point.

16. Write A, B, C, D, E, F for the coordinates in $\mathbb{P}^5$. Find the set of singular points on $V(AB - CD + EF, F)$. (This is a Schubert variety in the Grassmannian $Gr(2, 4)$, consisting of all planes that meet a fixed plane in more than a point).

17. A set is locally closed if it is the intersection of a closed subset with an open subset. A set is constructible if it is the union of a finite number of locally closed subsets. Chevalley's Theorem states that if $f : X \to Y$ is a morphism of algebraic varieties (over an algebraically closed field, though there are more general scheme theoretic versions), and if C is a constructible subset of X, then $f(C)$ is a constructible subset of Y.

    Consider $f : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$, $f(x, y) = (x, xy)$. Show that $f(\mathbb{A}^2)$ is not closed, not open, but is constructible.

18. Let R be an integral domain and K its field of fractions. Let $f \in R[t]$. If f can be factored in $K[t]$, show that there is a nonzero $r \in R$ such that rf can be factored in $R[t]$.

19. Let Y be an irreducible variety and $\pi : Y \times \mathbb{A}^1 \longrightarrow Y$ be the projection onto the first factor. Let C be a constructible subset of $Y \times \mathbb{A}^1$. Show that if $\pi(C)$ is dense in Y then $\pi(C)$ contains an open subset of Y.

    Hints:

    - First reduce to the case where C is the intersection of an open and a closed subset
    - Let R be the ring of regular functions on Y. Consider first the case where C is given by $f(t) = 0, g(t) \neq 0$ for some $f(t), g(t) \in R[t]$ which have no common factor in $K[t]$. Your open subset could be the subset of Y where a leading coefficient does not vanish and a resultant does not vanish.
    - Now consider the case where $f(t)$ and $g(t)$ have a greatest common factor $d(t)$ in $K[t]$. Use the result of the previous problem to show that we can lift this factorisation to $R[t]$ at the cost of passing to an open subset of Y. Replace $f(t)$ by $f(t)/d(t)$ and declare victory by induction on the degree of f.
    - In general, C is given by $f_1(t) = \cdots = f_r(t) = 0$, $g(t) \neq 0$ (why?). Again at the cost of passing to an open subset of Y, replace $f_1, \ldots, f_r$ by their greatest common divisor in $K[t]$ to reduce to the previous case.

20. Prove Chevalley's theorem (hint: it can be reduced to Problem 19).