Homework 1

Due: Thursday April 11, 2013

The notations $\Re(z)$ and $\Im(z)$ signify the real and imaginary parts of a complex number respectively.

1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

   (a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$

   (b) $1/z = \bar{z}$

   (c) $\Re(z) = 3$

   (d) $\Im(z) \geq c$ where $c \in \mathbb{R}$

   (e) $\Re(az + b) > 0$ where $a, b, \in \mathbb{C}$

   (f) $|z| = \Re(z) + 1$.

2. Let $w = re^{i\phi}$ where $r \geq 0$ and $\phi \in \mathbb{R}$. Give a complete description of all of the solutions to the equation $z^n = w$ where n is any integer.

3. Let $z = re^{i\theta}$ where $r > 0$ and $-\pi < \theta < \pi$ are real numbers. Define the logarithm function by

   $$\log(z) = \log(r) + i\theta.$$

   Show that this defines a holomorphic function on this domain of definition.

4. Let f be a power series centred at the origin. Prove that f has a power series expansion about any point in its disc of convergence.

5. The fundamental theorem of algebra states that if $P(z)$ is a nonconstant polynomial with coefficients in $\mathbb{C}$, then there exists $w \in \mathbb{C}$ with $P(w) = 0$. Give a proof of the fundamental theorem of algebra along the following lines:

   Using theorems of real analysis, show that the function $z \mapsto |P(z)|$ achieves a minimum on $\mathbb{C}$, say at the point z_0. Then use a Taylor series expansion of $P(z)$ about the point z_0 to arrive at a contradiction if $P(z_0) \neq 0$.