Q1: First we need to check that if $z \in \mathcal{H}$, then $f(z) \in \mathbb{D}$.

Write $z = x+iy$. For $z \in \mathcal{H}$, $y > 0$.

Then $|z-i|^2 = x^2+(y-1)^2 = x^2+y^2+1-2y$.

$|z+i|^2 = x^2+(y+1)^2 = x^2+y^2+1+2y$.

For $y>0$, $|z-i|^2 < |z+i|^2 \Rightarrow |z-i| < |z+i|$

$\Rightarrow \left|\frac{z-i}{z+i}\right| < 1$ as required.

Injectivity of f:

if $f(z_1) = f(z_2)$:

$$\frac{z_1-i}{z_1+i} = \frac{z_2-i}{z_2+i}$$

$$(z_1-i)(z_2+i) = (z_2-i)(z_1+i)$$

$$\cancel{z_1z_2} - iz_2 + iz_1 \cancel{+1} = \cancel{z_1z_2} - iz_1 + iz_2 \cancel{+1}$$

$$\therefore \quad i(z_1-z_2) = -i(z_1-z_2)$$

$$\therefore \quad z_1 - z_2 = 0 \Rightarrow z_1 = z_2.$$

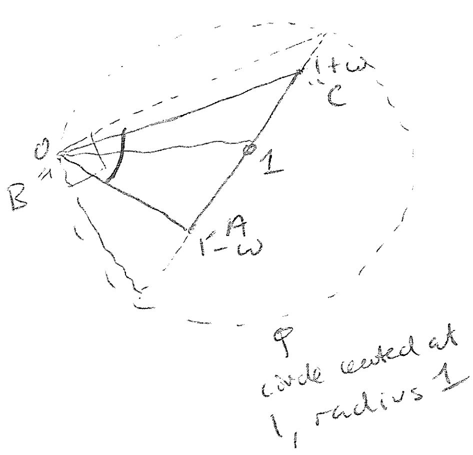
Surjectivity of f:

Let $w \in \mathbb{D}$. Let $z = -i \cdot \frac{w+1}{w-1} = i \cdot \frac{1+w}{1-w}$.

Then $$f(z) = \frac{i \cdot \frac{1+w}{1-w} - i}{i \cdot \frac{1+w}{1-w} + i} = \frac{\frac{1+w}{1-w} - 1}{\frac{1+w}{1-w} + 1} = \frac{1+w-(1-w)}{1+w+(1-w)} = \frac{2w}{2} = w.$$

So provided $z \in \mathcal{H}$ we've shown f is surjective.

To show $z \in \mathcal{H}$ we need to show $\mathrm{Re}\left(\frac{1+w}{1-w}\right) > 0$, ie $-\pi/2 < \arg\left(\frac{1+w}{1-w}\right) < \pi/2$.



The line segment between $1+w$ and $1-w$ has centre 1 and lies entirely within the circle centred at 1 with radius 1 as $|w|<1$.

Thus the angle ABC is acute as it is smaller than the $90°$ angle subtended by the diameter.

$\therefore -\pi/2 < \arg\left(\frac{1+w}{1-w}\right) < \pi/2$ as required.

The previous computations show that the inverse of f is given by $f^{-1}(\omega) = i \cdot \frac{1+\omega}{1-\omega}$.

$i \cdot \frac{1+\omega}{1-\omega}$ is a rational function with a pole at 1, which is not on $\mathbb{D}$.

$\therefore$ f^{-1} is a holomorphic function on $\mathbb{D}$.

Q2 (a): Let r be a real number greater than 1 and $\dfrac{\sum_{i=0}^{n-1} |a_i|}{|a_n|}$.

Then for $|z| = r$, we have

$$|q(z)| \le \sum_{i=0}^{n-1} |a_i| \, r^i \qquad \text{(by the triangle inequality)}$$

$$\le r^{n-1} \sum_{i=0}^{n-1} |a_i| \qquad \text{as } r > 1$$

$$< |a_n| \cdot r^n \qquad \text{as } r > \frac{\sum_{i=0}^{n-1} |a_i|}{|a_n|}.$$

$$= |a_n z^n| \qquad \text{, as required.}$$

Q2 (b): In part (a) we showed that the functions $q(z)$ and $a_n z^n$ satisfy $|q(z)| < |a_n z^n|$ on the circle with $|z| = r$.

Both the functions $q(z)$ and $a_n z^n$ are polynomials, hence are entire.

Therefore, by Rouche's theorem, the functions ~~q(z) and~~ $q(z) + a_n z^n$ and $a_n z^n$ have the same number of zeros inside the disc $\{z \in \mathbb{C} \mid |z| < r\}$, counted with multiplicity.

The function $a_n z^n$ has a zero at $z = 0$.

$\therefore$ The function $q(z) + a_n z^n$ must have a zero.

ie $p(z)$ has a zero, proving the fundamental theorem of algebra.

Q3. The set $X = \{ z \in \mathbb{C} \mid 0 \le \mathrm{Re}(z) \le 1 ,\ 0 \le \mathrm{Im}(z) \le |\mathrm{Im}(\tau)| \}$

is a closed and bounded subset of $\mathbb{C}$, hence is compact.

As f is continuous, $|f(z)|$ is bounded on X.

ie $\exists\, M \in \mathbb{R}$ such that $|f(z)| < M$ for all $z \in X$.

Now let $z \in \mathbb{C}$ be arbitrary. As $\mathrm{Im}(\tau) \neq 0$, there exists an integer n such that $z' = z - n\tau$ satisfies

$$0 \le \mathrm{Im}(z') \le |\mathrm{Im}(\tau)| .$$

There exists an integer m such that $z'' = z' - m$ satisfies

$$0 \le \mathrm{Re}(z'') \le 1 .$$

As $m \in \mathbb{R}$, $0 \le \mathrm{Im}(z'') \le |\mathrm{Im}(\tau)|$, so $z'' \in X$.

We have

$$\begin{aligned} f(z) &= f(z') && (\text{iterating } f(z+\tau) = f(z)) \\ &= f(z'') && (\text{iterating } f(z+1) = f(z)) \end{aligned}$$

$$\therefore\ |f(z)| = |f(z'')| < M \quad \text{as } z'' \in X .$$

Thus f is an entire, bounded function. By Louiville's Theorem, f is constant.

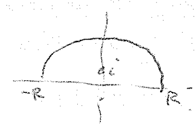
Q4: Note that $\cos(x) = \mathrm{Re}\left(e^{ix}\right)$.

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2}\,dx = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx = \mathrm{Re}\lim_{R\to\infty}\int_{-R}^{R} \frac{e^{ix}}{1+x^2}\,dx.$$

Consider the function $f(z) = \dfrac{e^{iz}}{1+z^2}$. It has two poles, at i and $-i$. The residue at $z=i$ is

$$\mathrm{res}_{z=i} f(z) = \lim_{z\to i} \frac{e^{iz}}{1+z^2}(z-i) = \lim_{z\to i}\frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}.$$

Let C_R be the semicircular arc $\{ z \in \mathbb{C} \mid |z| = R,\ \mathrm{Im}(z) \geq 0\}$



If $z \in C_R$ then $\mathrm{Im}(z) \geq 0 \Rightarrow \mathrm{Re}(iz) \leq 0 \Rightarrow \left|e^{iz}\right| \leq 1$.

Also $\left|\cancel{z}1+z^2\right| \geq \cancel{\leq} |z^2| - 1 = R^2-1$ (triangle inequality).

$$\therefore \left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{R^2-1}.$$

$$\therefore \left|\int_{C_R} \frac{e^{iz}}{1+z^2}\,dz\right| \leq \pi R \cdot \frac{1}{R^2-1}$$ as the length of C_R is πR.

As $\lim_{R\to\infty} \pi R \cdot \frac{1}{R^2-1} = 0$ this implies $\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{1+z^2}\,dz = 0$.

By the residue formula, for $R>1$:

$$\int_{C_R}\frac{e^{iz}}{1+z^2}\,dz + \int_{-R}^{R}\frac{e^{iz}}{1+z^2}\,dz = 2\pi i\, \mathrm{res}_{z=i} f(z)$$
$$= \frac{\pi}{e}.$$

Taking the limit as $R\to\infty$ yields $\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{iz}}{1+z^2}\,dz = \frac{\pi}{e}$

and hence $$\int_{-\infty}^{\infty}\frac{\cos(x)}{1+x^2}\,dx = \frac{\pi}{e}.$$

Q5 (a) Let K be a compact subset of $\mathbb{D}$.

Let $r = \sup_{z \in K} |z|$.

As K is compact and the absolute value function is continuous, there exists $w \in K$ with $|w| = r$. $\therefore\ r < 1$ (as $w \in \mathbb{D}$).

Choose R such that $r < R < 1$.

By Cauchy's inequality

$$\left| f_n^{(k)}(0) \right| \leq \frac{k!\, B}{R^k}.$$

Let $a_k = \lim_{n \to \infty} f_n^{(k)}(0)$.

Taking the limit in the above inequality implies $|a_k| \leq \dfrac{k!\, B}{R^k}$.

Let $f(z) = \sum_{k=0}^{\infty} \frac{a_k}{k!} . z^k$. We will show that $\{f_n\}$ converges uniformly to f on K.

We know $f_n(z) = \sum_{k=0}^{\infty} \dfrac{f_n^{(k)}(0)}{k!} z^k$.

For $z \in K$:

$$|f_n(z) - f(z)| = \left| \sum_{k=0}^{\infty} \frac{f_n^{(k)}(0) - a_k}{k!} z^k \right|$$

$$\leq \sum_{k=0}^{\infty} \frac{|f_n^{(k)}(0) - a_k|}{k!} |z|^k$$

$$\leq \sum_{k=0}^{\infty} |f_n^{(k)}(0) - a_k| \frac{r^k}{k!}$$

$$\leq \sum_{k=0}^{N-1} |f_n^{(k)}(0) - a_k| \frac{r^k}{k!} + \sum_{k=N}^{\infty} \left(|f_n^{(k)}(0)| + |a_k| \right) \frac{r^k}{k!}$$

$$\leq \sum_{k=0}^{N-1} |f_n^{(k)}(0) - a_k| \frac{r^k}{k!} + \sum_{k=N}^{\infty} \frac{2\, k!\, . B}{R^k} . \frac{r^k}{k!}$$

$$= \sum_{k=0}^{N-1} |f_n^{(k)}(0) - a_k| \frac{r^k}{k!} + 2B \left(\frac{r}{R} \right)^N \frac{1}{1 - \frac{r}{R}}.$$

Pick $\varepsilon > 0$. Then as $\frac{r}{R} < 1$, there exists $N \in \mathbb{N}$ such that

~~for $N \geq N_0$~~ $$2B\left(\frac{r}{R}\right)^N \frac{1}{1-\frac{r}{R}} < \frac{\varepsilon}{2}.$$ Fix this choice of N.

For each k with $0 \leq k < N$, the sequence $f_n^{(k)}(0)$ converges to a_k.

$\therefore$ There exists N_k such that for $n > N_k$,

$$\left| f_n^{(k)}(0) - a_k \right| < \varepsilon \cdot \frac{k!}{2N r^k}.$$

Then for $n > \max\{N_0, N_1, \ldots, N_{N-1}\}$ we have, from the inequality on the previous page:

$$|f_n(z) - f(z)| \leq \left(\sum_{k=1}^{N} \varepsilon \cdot \frac{k!}{2N r^k} \cdot \frac{r^k}{k!} \right) + \varepsilon/2$$

$$= \varepsilon.$$

This is true for all $z \in K$, so we have shown the desired uniform convergence.

(b) Let $f_n(z) = 3^n z^n$.

Then $$f_n^{(k)}(0) = \begin{cases} 0 & \text{if } k \neq n \\ 3^n . n! & \text{if } k = n. \end{cases}$$

$\therefore$ For each k, $\{f_n^{(k)}(0)\}_{n=0}^{\infty}$ is convergent, as it is eventually constant.

The sequence $\{f_n\}$ does not converge as

$$f_n(1/2) = (3/2)^n$$

which tends to ∞ as $n \to \infty$ (which also shows condition (i) does not hold).