Homework 6

Due: Thursday May 23, 2013

1. Define the function
$$\theta(x) = \sum_{p \le x} \log p$$
where the sum is over prime numbers p. Prove the identity
$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2} dt.$$

2. Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ and let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a real matrix with $\det A > 0$. Prove that the function
$$\rho_A(z) = \frac{az+b}{cz+d}$$
is a bijection from $\mathfrak{h}$ to $\mathfrak{h}$.

3. Let D be a disc with boundary C and Ω an open set containing D. Let f_n be a sequence of holomorphic functions converging uniformly on compact sets to a function f. Suppose that f has no zeros on C. Prove that there is an integer N such that for $n > N$, f and f_n have the same number of zeros in the disc D, counted according to multiplicity.

4. Prove the identity
$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \frac{x - \lfloor x \rfloor}{x^{s+1}} dx.$$
For which values of s is the right hand side of this equation absolutely convergent?