Solutions to 116 Homework 3

1. Define analytic continuations

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

these are both entire by inspection, and agree with the usual cos, sin on the real axis. We write $\cot(z) = \frac{\cos(z)}{\sin(z)}$.

Then cot is a quotient of entire functions. I claim that the zeros of analytic cos are disjoint from the zeros of analytic sin. Given this, cot has a zero at z_0 iff $\cos(z_0) = 0$. And cot has a pole at z_0 iff $1/\cot$ has a zero at z_0 iff $\sin(z_0) = 0$.

We calculate the zeros. Let $z_0 = a + ib$. Then $\cos(z_0) = 0$ iff $e^{2ia-2b} = -1$ iff $b = 0$ and $e^{2ia} = -1$. So zeros of analytic cosine are given by $\{\pi/2 + \pi n : n \in \mathbb{Z}\}$.

Similarly, the zeros of analytic sine are $\{\pi n : n \in \mathbb{Z}\}$.

The zero sets are disjoint, as claimed. Therefore the zeros of cot are $\{\pi/2 + \pi n : n \in \mathbb{Z}\}$, and the poles of cot are $\{\pi n : n \in \mathbb{Z}\}$.

We prove that every zero and pole is simple. It suffices to show that every zero of analytic cos, sin is simple. For this, we show that if $\cos(z_0) = 0$, then $\frac{d}{dz}\cos(z_0) \neq 0$, and the same for sin.

We have

$$\frac{d}{dz}\cos(z) = \frac{ie^{iz} - ie^{-iz}}{2} = -\sin(z)$$
$$\frac{d}{dz}\sin(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \cos(z)$$

and the result follows by disjointness of the zero sets.

2. f has at most a simple pole at z_0, so we can write

$$f(z) = \frac{a}{z - z_0} + g(z)$$

for $g(z)$ a holomorphic function, and $a = \text{Res}_{z_0} f$.

Choose M so that $|g| < M$ on the unit ball $B_1(z_0)$. Recall that the arclength $L(C_\epsilon) = \epsilon\theta$. Then

$$\left|\lim_{\epsilon \to 0} \int_{C_\epsilon} g(z)dz\right| = \lim_{\epsilon \to 0}\left|\int_{C_\epsilon} g(z)dz\right| \leq \lim_{\epsilon \to 0} M * \theta\epsilon = 0 \quad (1)$$

Let C_ϵ be parameterized by the curve

$$z(t) = z_0 + \varepsilon e^{i\phi}, \quad \phi \in [\alpha, \alpha + \theta]$$

for some $\alpha \in [0, 2\pi)$. Then

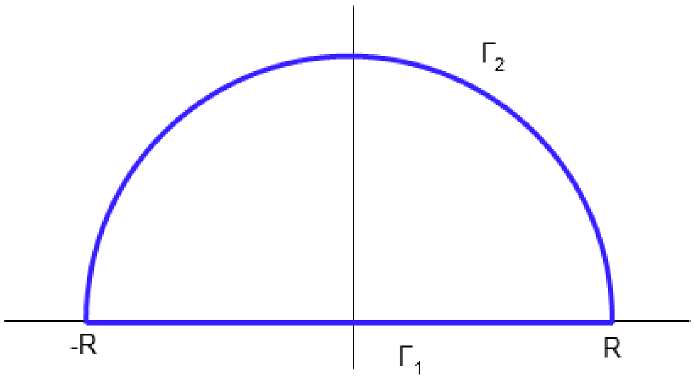
$$\int_{C_\epsilon} \frac{a}{z - z_0} dz = \int_{\alpha}^{\alpha+\theta} \frac{a}{\epsilon e^{i\phi}} i\epsilon e^{i\phi} d\phi = i\theta a \tag{2}$$

Combining (1) and (2), we have that

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f dz = \lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{a}{z - z_0} dz = i\theta \text{Res}_{z_0} f$$

recalling that $a = \text{Res}_{z_0} f$.

3. Let Γ be the contour oriented counter-clockwise



Define the meromorphic function $f(z) = \frac{1}{1+z^4}$. We note that

$$\lim_{R \to \infty} \left| \int_{\Gamma_2} f dz \right| \leq \lim_{R \to \infty} \frac{\pi R}{R^4 - 1} = 0$$

So

$$\lim_{R \to \infty} \int_{\Gamma} f dz = \lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1 + x^4} dx = 2 \int_{0}^{\infty} \frac{1}{1 + x^4} \tag{3}$$

where we also use the fact that $\frac{1}{1+x^4}$ is even.

Let

$$z_k = e^{i\pi(1/4 + k/2)}, \quad k = 1, 2, 3, 4$$

be the roots of $1 + z^4$. Then f has a simple pole at each z_k. In particular, the contour Γ contains the poles z_1, z_2.

We calculate

$$\text{Res}_{z_1} f = \lim_{z \to z_1} (z - z_1) \frac{1}{1 + z^4} = e^{-3i\pi}/4$$

and similarly $\mathrm{Res}_{z_2} f = e^{-i\pi}/4$.

So for R is sufficiently large, the residue theorem says that

$$\int_\Gamma f dz = 2\pi i(\mathrm{Res}_{z_1} f + \mathrm{Res}_{z_2} f) = \pi/\sqrt{2} \tag{4}$$

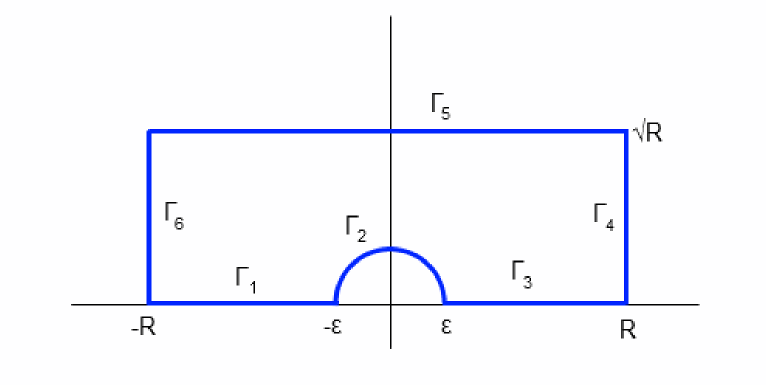
Combining (3) and (4) we have

$$\int_0^\infty \frac{1}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}$$

4. If $R \in \mathbb{R}_+$, then since $\frac{\sin(x)}{x}$ is bounded we can write

$$\int_0^R \frac{\sin(x)}{x} dx = \lim_{\epsilon\to 0} \int_\epsilon^R \frac{\sin(x)}{x} dx$$

Let Γ be the contour (oriented counter-clockwise)



Define the function $f(z) = e^{iz}/z$, so f is meromorphic with a simple pole at 0. Observe that

$$\begin{aligned}\int_{\Gamma_1 \cup \Gamma_3} f dz &= \int_{-R}^{-\epsilon} \frac{\cos(x) + i\sin(x)}{x} dz + \int_\epsilon^R \frac{\cos(x) + i\sin(x)}{x} dx \\ &= 2i \int_\epsilon^R \frac{\sin(x)}{x}\end{aligned}$$

So by our initial remark

$$\lim_{R\to\infty} \lim_{\epsilon\to\infty} \int_{\Gamma_1 \cup \Gamma_3} f dz = 2i \int_0^\infty \frac{\sin(x)}{x} dx$$

We determine these limits for other components of Γ. Note that Γ_4, Γ_5, Γ_6 are independent of ϵ, while Γ_2 is independent of R.

(calculate Γ_2*)* f has a simple pole at 0, and Γ_2 is a (clockwise-oriented!) half-circle of radius ϵ and centered at 0, so by Q2

$$\lim_{\epsilon\to 0}\int_{\Gamma_2} f dz = -i\pi \mathrm{Res}_0 f = -i\pi$$

(calculate Γ_4*)*

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_4} f dz\right| &\leq \lim_{R\to\infty}\int_0^{\sqrt{R}}\left|\frac{e^{iR-t}}{R+it}\right| dt\\ &\leq \lim_{R\to\infty}\frac{1}{R} * \sqrt{R}\\ &= 0\end{aligned}$$

(calculate Γ_5*)*

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{\Gamma_5} f dz\right| &\leq \lim_{R\to\infty}\int_{-R}^{R}\left|\frac{e^{it-\sqrt{R}}}{t+i\sqrt{R}}\right| dt\\ &\leq \lim_{R\to\infty}\frac{e^{-\sqrt{R}}}{\sqrt{R}} * 2R\\ &= 0\end{aligned}$$

(calculate Γ_6*)* same as Γ_4

Now trivially f is holomorphic in $\mathbb{C} - \{0\}$, so $\int_\Gamma f = 0$ for every non-zero R, ϵ. Combining the above calculations shows that

$$0 = \lim_{R\to\infty}\lim_{\epsilon\to 0}\int_\Gamma f dz = -i\pi + 2i\int_0^\infty \frac{\sin(x)}{x}dx$$

and hence

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$