Solutions to 116 Homework 4

1. If f is holomorphic and non-zero in some region U, then both f' and $1/f$ are holomorphic in U. So F can only have poles where f has a zero or pole.

Let z_0 be a zero or pole of f. Then since f is meromorphic, near z_0 we can write

$$f = (z - z_0)^k g(z)$$

for k a non-zero integer, and g some holomorphic function such that $g(z_0) \neq 0$. Then

$$f' = k(z - z_0)^{k-1} g + (z - z_0)^k g'$$

So near z_0

$$F = \frac{k}{z - z_0} + \frac{g'}{g}$$

and since g is holomorphic, non-zero near z_0, the quotient g'/g is holomorphic. Therefore F has a simple pole at z_0 with residue k.

In particular, if z_0 is a zero of order k, then F has residue k. And if z_0 is a pole of order k, then F has residue $-k$.

2. Let $f(z) = \frac{1}{2-e^z}$, so $b_n = f^{(n)}(0)$. Trivially f has simple poles at $\log 2 + 2\pi i n$ ($n \in \mathbb{Z}$), each with residue $-1/2$. So the function

$$g(z) = \frac{1}{2(z - \log 2)} + f(z)$$

is holomorphic on the ball $B_5(0)$ of radius 5 centered at 0 (in fact, up to radius $\sqrt{(\log 2)^2 + 4\pi^2}$). We can write g as a power series $g = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence at least 5.

Now

$$f^{(n)}(0) = \frac{n!}{2(\log 2)^{n+1}} + n! a_n$$

But since $\log 2 < 5$ is in the radius of convergence of g, we have that

$$\lim_{n\to\infty} \frac{n! a_n}{\frac{n!}{2(\log 2)^{n+1}}} = \log 2 \lim_{n\to\infty} a_n (\log 2)^n = 0$$

and hence $f^{(n)}(0) \sim \frac{n!}{2(\log 2)^{n+1}}$.

3. Our generating function is entire, so by the Cauchy integral formula we have

$$t_n = \frac{n!}{2\pi i}\int_C \frac{e^{z+z^2/2}}{z^{n+1}}dz$$

where C is any circle centered at 0. Let C have radius r, and we can write

$$t_n = \frac{n!}{2\pi r^n}\int_{-\pi}^{\pi} \exp(re^{i\theta} + r^2/2e^{2i\theta} - in\theta)d\theta$$

We wish to use Hayman's method. In particular, we seek an $\epsilon(n)$ and $r(n)$ so that (as functions of n)

$$\int_{-\pi}^{\pi} \exp(re^{i\theta} + r^2/2e^{2i\theta} - in\theta)d\theta \sim \int_{-\epsilon}^{\epsilon} \exp(re^{i\theta} + r^2/2e^{2i\theta} - in\theta)d\theta \quad (1)$$

$$\sim \int_{-\epsilon}^{\epsilon} \exp(r + r^2/2 - \theta^2 F(n))d\theta \quad (2)$$

For $F(n)$ some function of n. In essence we want r so that in the Taylor expansion of $re^{i\theta} + r^2/2e^{2i\theta} - in\theta$, the coefficient of θ is 0. Therefore we must have $r + r^2 - n = 0$, and hence

$$r(n) = \sqrt{1/4 + n} - 1/2 \quad (3)$$

This means that $F(n) = r/2 + r^2$. Using the change of variables $y = \theta\sqrt{r/2 + r^2}$

$$\begin{aligned}\sqrt{r/2 + r^2}\int_{-\epsilon}^{\epsilon} \exp(-\theta^2 F(n))d\theta &= \int_{-\epsilon\sqrt{O(n)}}^{\epsilon\sqrt{O(n)}} e^{-y^2}dy \\ &\sim \int_{-\infty}^{\infty} e^{-y^2}dy \\ &= \sqrt{\pi}\end{aligned}$$

Hence, denoting the integral (2) by I,

$$I \sim \frac{e^{r+r^2/2}}{\sqrt{r/2 + r^2}}\sqrt{\pi} \sim \frac{e^{n/2+\sqrt{n}/2-1/4}}{\sqrt{n}}\sqrt{\pi}$$

having additionally replaced r by the expression (3).

An argument following the handout on Hayman's method shows we can take $\epsilon(n) = n^{-2/5}$, so that (1), (2) hold. In fact we can take $\epsilon = n^{\alpha}$, for any $\alpha \in (-1/3, -1/2)$. Then the above shows that

$$t_n \sim \frac{n!}{2\sqrt{\pi n}r^n}e^{n/2+\sqrt{n}/2-1/4} \sim \frac{n^n}{\sqrt{2}r^n}e^{-n/2+\sqrt{n}/2-1/4}$$

In the second relation we used Stirling's formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

We find an asymptotic expression for r^n. Recall the Taylor expansions $\sqrt{1+x} = 1 + 1/2x + O(x^2)$ and $\log(1-x) = -x - 1/2x^2 + O(x^3)$, for x sufficiently small. We have, for n sufficiently large,

$$\begin{aligned}
\log(r^n) &= n\log(\sqrt{1/4+n} - 1/2) \\
&= n/2\log(n) + n\log\left(\sqrt{\frac{1}{4n}+1} - \frac{1}{2\sqrt{n}}\right) \\
&= n/2\log(n) + n\log\left(1 + \frac{1}{8n} + O(n^{-2}) - \frac{1}{2\sqrt{n}}\right) \\
&= n/2\log(n) - n\left(-\frac{1}{8n} + \frac{1}{2\sqrt{n}}\right) - n/2\left(-\frac{1}{8n} + \frac{1}{2\sqrt{n}}\right)^2 + O(n^{-3/2}) \\
&= n/2\log(n) - \sqrt{n}/2 + O(n^{-1/2})
\end{aligned}$$

Hence

$$r^n \sim n^{n/2}e^{-\sqrt{n}/2}$$

and the result follows.

4. We first prove convergence. Write $\prod_{n=1}^{\infty}(1+z_n) = C\prod_{n=M}^{\infty}(1+z_n)$ where $|z_n| < 1/2$ for every $n \geq M$. Clearly it suffices to prove convergence under the assumption that $|z_n| < 1/2$ for every n.

We can define the complex logarithm on the ball $B_{1/2}(1)$, and by our assumption $1 + z_n \in B_{1/2}(1)$ for each n. It therefore suffices to show that the sum $\sum_{n=1}^{\infty}\log(1+z_n)$ exists, since by continuity

$$\begin{aligned}
\exp\left(\lim_{N\to\infty}\sum_{n=1}^{N}\log(1+z_n)\right) &= \lim_{N\to\infty}\exp\left(\sum_{n=1}^{N}\log(1+z_n)\right) \\
&= \lim_{N\to\infty}\prod_{n=1}^{N}(1+z_n)
\end{aligned}$$

Further, recall the power series expansion $\log(1+z) = z - 1/2z^2 + 1/3z^3 + \ldots$, converging for all $|z| < 1$. Therefore

$$|\log(1+z_n)| \leq |z_n| + |z_n|\sum_{n=1}^{\infty}|z_n|^n \leq 2|z_n|$$

So $\sum_{n=1}^{\infty} \log(1+z_n)$ is absolutely convergent, and hence convergent. So by the above discussion the infinite product $\prod_{n=1}^{\infty}(1+z_n)$ converges.

Now if every $z_n \neq -1$, then $C \neq 0$ and we can write $\prod_{n=M}^{\infty}(1+z_n) = e^{\Sigma}$, for $\Sigma \in \mathbb{C}$. So the product converges to a non-zero limit. Conversely, if $z_n = -1$ for some n, then necessarily $n < M$, and so $C = 0$.