Solutions to 116 Homework 5

1. Recall by Hadamard's theorem we have the identity

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n=1}^{\infty} (1 + s/n) e^{-s/n}$$

where γ is Euler's constant.

Choose a small closed ball B on which Γ is holomorphic. Then on B we have

$$\log \Gamma(s) = -\gamma s - \log s + \sum_{n=1}^{\infty} -\log(1 + s/n) + \frac{s}{n}$$

This sum converges uniformly on B. Therefore we take derivatives and find (for $s \in B$)

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - s^{-1} + \sum_{n=1}^{\infty} \frac{-1}{n+s} + \frac{s}{n} \tag{1}$$

but both sides of (1) are meromorphic functions on a connected set, and agree on B, so identity (1) must hold everywhere.

Recall that $\Gamma(1) = 1$. Therefore

$$\begin{aligned}
\Gamma'(1) &= -\gamma - 1 + \sum_{n=1}^{\infty} \frac{-1}{1+n} + \frac{1}{n} \\
&= -\gamma - 1 + \lim_{N \to \infty} \left(1 - \frac{1}{1+N} \right) \\
&= -\gamma
\end{aligned}$$

2. **A.** Using identities of Γ, for $z \notin \mathbb{Z}$,

$$\frac{\pi}{\sin \pi z} = \Gamma(z)\Gamma(1-z) = (-z)\Gamma(z)\Gamma(-z) \tag{2}$$

We calculate

$$\begin{aligned}
\Gamma(z)\Gamma(-z) &= e^{-\gamma z} \frac{1}{z} \left(\prod_{n=1}^{\infty} \frac{e^{z/n}}{1 + z/n} \right) e^{\gamma z} \frac{-1}{z} \left(\prod_{n=1}^{\infty} \frac{e^{-z/n}}{1 - z/n} \right) \\
&= \frac{-1}{z^2} \prod_{n=1}^{\infty} \left(\frac{e^{z/n}}{1 + z/n} \right) \left(\frac{e^{-z/n}}{1 - z/n} \right) \\
&= \frac{-1}{z^2} \prod_{n=1}^{\infty} \frac{1}{1 - z^2/n^2}
\end{aligned} \tag{3}$$

where we are justified in rearranging the products since the sums

$$\left(\sum_{n=1}^{\infty} -\log(1+z/n)+z/n\right)+\left(\sum_{n=1}^{\infty} -\log(1-z/n)-z/n\right)$$

are absolutely convergent. Combine (2) with (3) and the result follows.

B. We show $\sin z$ has growth order 1. On the one hand, writing $z=x+iy$, we have

$$|\sin z| = \left|\frac{e^{iz}-e^{-iz}}{2i}\right| \leq \frac{1}{2}(e^{-y}+e^{y}) \leq Ce^{y} \leq Ce^{|z|}$$

and so $\sin z$ has growth order ≤ 1. Conversely,

$$|\sin iy| = \left|\frac{e^{-y}-e^{y}}{2i}\right| \geq C'e^{y} = C'e^{|iy|}$$

The zeros of $\sin \pi z$ are the integers, and all have order 1. By Hadamard's theorem we can write

$$\begin{aligned}\sin \pi z &= e^{az+b} z \prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} (1-z/n)e^{z/n} \\ &= e^{az+b} z \prod_{n=1}^{\infty} (1-z/n)e^{z/n}(1+z/n)e^{-z/n} \\ &= e^{az+b} z \prod_{n=1}^{\infty} (1-z^2/n^2) \qquad (4)\end{aligned}$$

for some constants a, b.

We determine a, b. Since sin is odd, and the infinite product in (4) is even, we must have

$$-e^{az+b}z = e^{-az+b}(-z)$$

and hence $a=0$.

To find b, recall that $\lim_{z\to 0} \frac{\sin z}{z} = 1$. So using (4)

$$1 = \lim_{z\to 0} \frac{\sin \pi z}{\pi z} = \frac{1}{\pi} e^{b}$$

and so $e^{b} = \pi$.

3. Let $\Re(s) = \sigma > 1$. Note that if $ab > N^2$, then necessarily $a > N$ or $b > N$. For any N, M we have

$$\begin{aligned}\left|\sum_{n=1}^{N^2} \frac{d(n)}{n^s} - \left(\sum_{n=1}^{M} \frac{1}{n^s}\right)^2\right| &= \left|\sum_{\substack{ab>N^2\\a,b\leq M}} \frac{1}{a^s b^s}\right| \\ &\leq \sum_{\substack{ab>N^2\\a,b\leq M}} \frac{1}{a^\sigma b^\sigma} \\ &\leq 2\sum_{b=1}^{M} \frac{1}{b^\sigma} \sum_{a=N}^{M} \frac{1}{a^\sigma} \\ &\leq 2\zeta(\sigma) \sum_{a=N}^{M} \frac{1}{a^\sigma}\end{aligned}$$

The right hand side is bounded by $2\zeta(\sigma)^2$. Taking $M \to \infty$, we have

$$\left|\sum_{n=1}^{N^2} \frac{d(n)}{n^s} - \zeta(s)\right| \leq 2\zeta(\sigma) \sum_{n=N}^{\infty} \frac{1}{n^\sigma}$$

But the right hand side tends to zero as $N \to \infty$, since the sum is convergent for every $\sigma > 1$. The result follows.

4. We first show that $\prod_p (1 - p^{-s})^{-1}$ defines a holomorphic function on $\Omega = \{\Re(s) > 1\}$. Let K be a compact subset of Ω. Then $\Re(s) > \sigma > 1$ on K.

If $s \in K$, then $|p^{-s}| < p^{-\sigma} < 1/2$ for $p > N$. So on K we have

$$\begin{aligned}\sum_p |(1 - p^{-s})^{-1} - 1| &\leq \sum_p p^{-\sigma} |1 - p^{-s}|^{-1} \\ &\leq C + \sum_{p>N} 2p^{-\sigma} \\ &\leq C + 2\zeta(\sigma)\end{aligned}$$

So by Homework 4 the product $\prod_p (1 - p^{-s})^{-1}$ converges uniformly on compact sets, and hence defines a holomorphic function on Ω. And since $(1 - p^{-s})^{-1} \neq 0$ for every $s \in \Omega$, the convergent is non-vanishing.

By analytic continuation it now suffices to prove the identity when $s > 1$. Recall that

$$\frac{1}{1 - p^{-s}} = 1 + p^{-s} + p^{-2s} + \dots \tag{5}$$

By the fundamental theorem of arithmetic, for every N and M we have

$$\prod_{p \leq N} (1 + p^{-s} + p^{-2s} + \ldots + p^{-Ms}) \leq \sum_{n=1}^{N!^M} \frac{1}{n^s} \leq \zeta(s)$$

where the product ranges over primes $\leq N$.

By identity (5), we can take $M \to \infty$, and obtain

$$\prod_{p \leq N} (1 - p^{-s})^{-1} \leq \zeta(s)$$

for every N. Taking $N \to \infty$,

$$\prod_{p} (1 - p^{-s})^{-1} \leq \zeta(s)$$

Conversely,

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n^s} &\leq \prod_{p \leq N} (1 + p^{-s} + \ldots + p^{-Ns}) \\ &\leq \prod_{p \leq N} (1 - p^{-s})^{-1} \\ &\leq \prod_{p} (1 - p^{-1})^{-1} \end{aligned}$$

and the result follows by taking $N \to \infty$.

We show that $\zeta(-3 + 47i) \neq 0$. Recall the identity $\xi(s) = \xi(1 - s)$, which can be written

$$\pi^{-s/2} \Gamma(s/2) \zeta(s) = \pi^{-(1-s)/2} \Gamma((1 - s)/2) \zeta(1 - s) \tag{6}$$

The Γ function is never zero, nor is any exponent of π. By the above discussion $\zeta(1 - (-3 + 47i)) \neq 0$, and so (6) shows that $\zeta(-3 + 47i) \neq 0$.