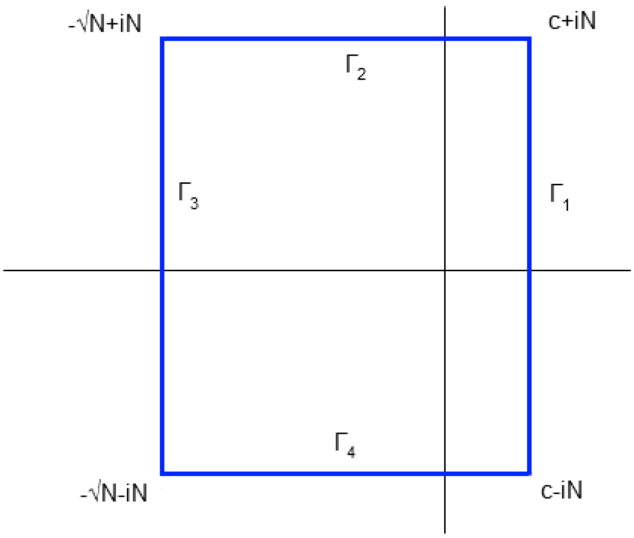
Solutions to 116 Homework 7

1. By definition, $i^i = e^{i \log i}$. Depending on which branch we pick, $\log i = i(\pi/2 + 2\pi k)$ for $k \in \mathbb{Z}$. So the possible values for i^i are

$$\{e^{-\pi/2+2\pi k} : k \in \mathbb{Z}\}$$

2. First suppose $a > 1$, and let Γ be the contour oriented counter-clockwise



Since $c > 0$, we have for all N sufficiently large

$$\int_\Gamma \frac{a^s}{s} ds = 2\pi i \operatorname{Res}_{s=0} \frac{a^s}{s} = 2\pi i$$

We wish to prove that the contour integrals over Γ_2, Γ_3 and Γ_4 vanish as N tends to infinity. This will imply that

$$\lim_{N\to\infty} \int_\Gamma \frac{a^s}{s} ds = \lim_{N\to\infty} \int_{\Gamma_1} \frac{a^s}{s} ds = \lim_{N\to\infty} \int_{c-iN}^{c+iN} \frac{a^s}{s} ds$$

which proves the result for $a > 1$.

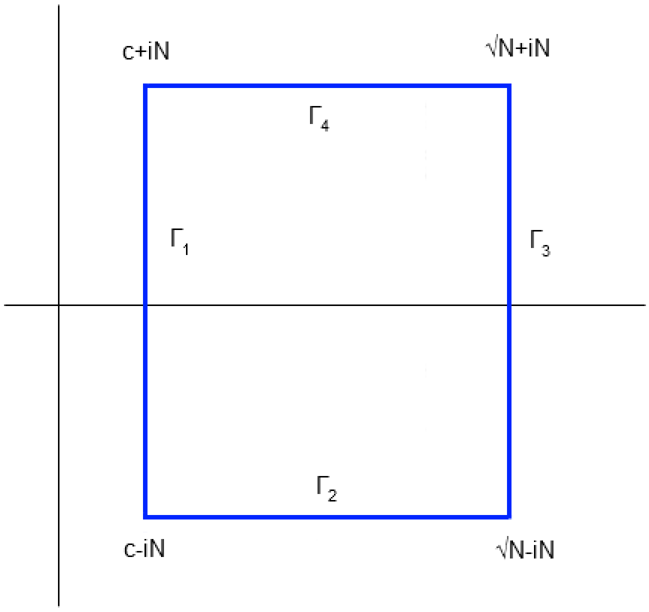
We calculate

$$\lim_{N\to\infty} \left| \int_{\Gamma_2 \cup \Gamma_4} \frac{a^s}{s} ds \right| \leq 2 \lim_{N\to\infty} \frac{a^c}{N} * (\sqrt{N} + c) = 0$$

And

$$\lim_{N\to\infty} \left| \int_{\Gamma_3} \frac{a^s}{s} ds \right| \leq \lim_{N\to\infty} \frac{a^{-\sqrt{N}}}{\sqrt{N}} * 2N = 0$$

Now take $a < 1$. We let Γ be the clockwise contour



By construction Γ contains no poles of $\frac{a^s}{s}$, so we have

$$\int_\Gamma \frac{a^s}{s} ds = 0$$

for every N.

As before it suffices to prove that, as $N \to \infty$, the integral tends to zero on the contours Γ_2, Γ_3 and Γ_4. We calculate

$$\lim_{N\to\infty} \left| \int_{\Gamma_2 \cup \Gamma_4} \frac{a^s}{s} ds \right| \leq 2 \lim_{N\to\infty} \frac{a^c}{N} * (\sqrt{N} - c) = 0$$

And

$$\lim_{N\to\infty} \left| \int_{\Gamma_3} \frac{a^s}{s} ds \right| \leq \lim_{N\to\infty} \frac{a^{\sqrt{N}}}{\sqrt{N}} * 2N = 0$$

3. Recall that $\zeta \neq 0$ on $\Re z \geq 1$. Let

$$\alpha = \min_{t \in [a,b]} |\zeta(1 + it)| > 0$$

Since ζ is holomorphic on $\mathbb{C} - \{1\}$, ζ is uniformly continuous on the compact set $K = [0, 1] + i[a, b]$. Choose δ so that if $z, w \in K$ and $|z - w| < \delta$, then $|\zeta(z) - \zeta(w)| < \alpha/2$. Then if $1 - \delta < \sigma \leq 1$, we have

$$|\zeta(\sigma + it)| \geq |\zeta(1 + it)| - |\zeta(1 + it) - \zeta(\sigma + it)| > \alpha/2$$

Hence $\zeta \neq 0$ on $(1 - \delta, \infty) + i(a, b)$.

4. We wish to show that

$$\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)}\zeta(s)ds = \sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)}\frac{1}{n^s}ds \tag{1}$$

This will follow by Fubini if we can show that both sides are absolutely convergent. In other words, we need

$$\int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \left|\frac{x^s}{s(s+1)}\frac{1}{n^s}ds\right| < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} \left|\frac{x^s}{s(s+1)}\frac{1}{n^s}ds\right| < \infty$$

Using that c is positive, we have

$$\left|\frac{x^s}{s(s+1)}\frac{1}{n^s}\right| = \frac{x^c}{n^c}\left|\frac{1}{s(s+1)}\right| \leq x^c \frac{1}{n^c}\frac{1}{|s|^2}$$

Therefore it suffices prove that the integral over the right hand side is absolutely convergent.

$$\begin{aligned} \int_{c-i\infty}^{c+i\infty} \left|\frac{1}{s^2}ds\right| &= \lim_{N\to\infty} \int_{-N}^{N} \frac{1}{c^2+t^2}dt \\ &= \lim_{N\to\infty} \frac{1}{c^2}\arctan(x/c)\Big|_{-N}^{N} \\ &= \frac{\pi}{c^2} \end{aligned}$$

And the claim follows.

Recall

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)}ds = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 - 1/x & \text{if } x \geq 1 \end{cases} \tag{2}$$

Combining (2) and (1), we have

$$\begin{aligned} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s(s+1)}\zeta(s)ds &= \sum_{n=1}^{\infty} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)}ds \\ &= 2\pi i \sum_{n\leq x} 1 - n/x \\ &= 2\pi i \lfloor x \rfloor \left(1 - \frac{\lfloor x \rfloor + 1}{2x}\right) \end{aligned}$$

We calculate the residue of the integrand of (1) at $s = 1$. Recall that $\zeta(s)$ has a simple pole at $s = 1$, with residue 1. And trivially $\frac{x^s}{s(s+1)}$ is holomorphic near 1. Therefore

$$\text{Res}_{s=1}\frac{x^s}{s(s+1)}\zeta(s) = \frac{x}{2}\text{Res}_{s=1}\zeta(s) = \frac{x}{2}$$

5. Let $K \subset \{\Re s > L\}$ be a compact set. Let $L + 2\epsilon$ be a minimum for $\Re(K)$, so that $\Re s \geq L + 2\epsilon > L$ for every $s \in K$. By assumption we can choose N so that

$$L + \epsilon \geq \sup_{n>N} 1 + \frac{\log |a_n|}{\log n}$$

So when $n > N$, $s \in K$ and $a_n \neq 0$ we have

$$\begin{aligned} |a_n n^{-s}| &\leq |a_n| n^{-L-2\epsilon} \\ &= n^{\frac{\log |a_n|}{\log n} - L - 2\epsilon} \\ &\leq n^{-1-\epsilon} \end{aligned}$$

Hence on K

$$\left| \sum_{n=N}^{\infty} a_n n^{-s} \right| \leq \sum_{n=N}^{\infty} n^{-1-\epsilon} < \infty$$

where ϵ is independent of s. So $\sum_{n=1}^{\infty} a_n n^{-s}$ is uniformly convergent on K.