Homework 1

Due: Thursday April 12, 2012

1. Here are five groups of order 6: C_6, $C_2 \times C_3$, $GL_2(\mathbb{F}_2)$, S_3, D_3. Determine which of these groups are isomorphic. (D_3 is the dihedral group of symmetries of an equilateral triangle).

2. For a permutation $\pi \in S_n$, we will define its support to be

$$\mathrm{supp}(\pi) = \{i \mid \pi(i) \neq i\}.$$

   Suppose that σ and τ are two permutations in S_n such that $\mathrm{supp}(\sigma) \cap \mathrm{supp}(\tau)$ contains exactly one element. Prove that the commutator $\sigma\tau\sigma^{-1}\tau^{-1}$ is a 3-cycle.

3. The *Heisenberg group* over a field F is

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in F \right\}.$$

   Let

$$X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}$$

   be elements of $H(F)$.

   (a) Compute the matrix product XY and deduce that $H(F)$ is closed under matrix multiplication. Exhibit explicit matrices such that $XY \neq YX$ (so that $H(F)$ is always non-abelian).

   (b) Find an explicit formula for X^{-1} and deduce that $H(F)$ is closed under taking inverses.

4. Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with the multiplication rules:

$$ij = k = -ij, \quad jk = i = -kj, \quad ki = j = -ik, \quad i^2 = j^2 = k^2 = -1,$$

   and 1 and -1 multiply as expected. This is a group, the quaternion group of order 8 (you do not have to prove this statement). Prove that Q_8 is not isomorphic to D_4.

5. Let k be a field. Show that $r \cdot (x, y) = (x + ry, y)$ defines an action of the additive group k on k^2.

6. Let $GL_n(\mathbb{Z})^{naive}$ be the set of all $n \times n$ matrices with entries in $\mathbb{Z}$ and non-zero determinant. Show that this does not form a group under matrix multiplication.

   If we define $GL_n(\mathbb{Z})$ to be the set of all $n \times n$ matrices with entries in $\mathbb{Z}$ whose determinant is a unit (i.e. $\det(A).t = 1$ for some integer t), show that $GL_n(\mathbb{Z})$ becomes a group under matrix multiplication.

7. If a and b are two commuting elements in a group G, prove that $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}$.

8. Shew that if $g^2 = 1$ for every element g of a group G, then G is abelian.

9. Given a permutation σ, write it (as usual) as a product of disjoint cycles. If σ has λ_i i-cycles for each i, we say that σ has *cycle type* $\lambda = (\lambda_1, \lambda_2, \ldots)$.

   Two elements a and b of a group G are said to be *conjugate* if there exists $g \in G$ such that $a = gbg^{-1}$. Prove that two elements of the symmetric group S_n are conjugate if and only if they have the same cycle type.

10. The *centre* of a group G consists of all elements z such that $zg = gz$ for all $g \in G$. For each n, find the centre of the dihedral group D_n.