Homework 2

Due: Thursday April 19, 2012

1. Find the size of the conjugacy class of the element $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ in $GL_2(\mathbb{F}_5)$. (The *conjugacy class* of $g \in G$ is the set of all elements of the form hgh^{-1} for some $h \in G$.)

2. Let G be a group. Prove that the map from G to itself given by $g \mapsto g^2$ is a homomorphism if and only if G is abelian.

3. Consider the set of necklaces with p beads, with each bead coloured in one of a different colours. The cyclic group of order p acts on this set by rotating the necklace. In this setup, show how the formula

$$\#\text{orbits} = \frac{1}{|G|} \sum_{g \in G} \text{Fix}(g)$$

yields a proof of Fermat's Little Theorem.

4. Let X be a set with 4 elements. Let Y be the set of unordered pairs of subsets $\{A, B\}$ of X with $|A| = |B| = 2$ and $A \cap B = \emptyset$. Then Y is a set with 3 elements. The symmetric group S_4 acts on X in the usual manner and this induces an action of S_4 on Y. Hence we have obtained a homomorphism from S_4 to S_3. Prove that this homomorphism is surjective and its kernel is isomorphic to $C_2 \times C_2$ (this is called the *Klein four* group).

5. Let G be an abelian group. Prove that the set $\{g \in G \mid g^n = 1 \text{ for some } n \geq 1\}$ is a subgroup of G (it goes by the name of the *torsion subgroup*). Give an example of a nonabelian group G where this statement fails.

6. Let H be a subgroup of G of index 2. Prove that H is a normal subgroup of G.

7. Find all possible subgroups of the alternating group A_4 and determine which are normal.

8. Let F be a field. Show that the determinant defines a surjective group homomorphism from $GL_n(F)$ to $F^\times$, with kernel $SL_n(F)$.