Answer all questions. All questions have equal value.

1. Let $F$ be a field.

   (a) Which field axiom tells us that $0 + 0 = 0$?

   (b) Prove that $F$ contains a unique zero element.

2. Find the infimum and the supremum of the following sets

   (a)
   $$S = \left\{ \frac{2}{n} \mid n \in \{1, 2, 3, 4, \ldots\} \right\}.$$

   (b)
   $$T = \bigcup_{n=1}^{\infty} [2n, 2n + 1].$$

3. Show that $|x - y| < \epsilon$ if and only if $x - \epsilon < y < x + \epsilon$.

4. Prove that if $x \in \mathbb{R}$, then there exists $n \in \mathbb{Z}$ satisfying $x \leq n < x + 1$.

   *[For this question you may NOT use the floor or ceiling functions ($\lfloor x \rfloor$ and $\lceil x \rceil$) since the purpose of this question is to prove that the ceiling function exists from first principles.]*

5. (a) Let $a$ be a positive real number with $a^2 < 2$. Let $b = \frac{2(a+1)}{a+2}$. Show that $a < b$ and $b^2 < 2$.

   (b) Let $A = \sup\{r \in \mathbb{R} \mid r > 0, r^2 < 2\}$. Apply the previous result with $a = A$ to show that $A^2 \geq 2$ *[Hint: proof by contradiction]*.