Answer all questions. All questions have equal value. Proofs are required unless explicitly stated otherwise.

1. Using the $\epsilon - N$ definition of a limit, prove that

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0.$$

2. Find

$$\lim_{n \to \infty} \frac{2n^2 + 10n + 5}{n^2}.$$

3. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences such that $\{y_n\}_{n=1}^{\infty}$ converges to zero. Suppose that for all positive integers k and m with $m \geq k$, we have

$$|x_m - x_k| \leq y_k.$$

Prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

4. Does the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

converge?

5. Recursively define a sequence by $x_1 = 1$ and

$$x_{n+1} = (\sqrt{2})^{x_n}.$$

Prove that the sequence $\{x_n\}_{n=1}^{\infty}$ converges.