Complex Numbers ($\mathbb{C}$)

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A complex number $z = (x, y) = x + iy$ where $i^2 = -1$.
$x = \text{Re}\,(z)$, the real part of z, $y = \text{Im}\,(z)$, the imaginary part of z.
$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$
$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$
$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$

Argand diagram
$x + iy$ is represented by the point (x, y) in the plane

Modulus/Argument (polar form)
$z = x + iy = r\cos\theta + i\sin\theta = r\,\text{cis}\,\theta = re^{i\theta}$ where
$r = \sqrt{x^2 + y^2} = |z|$, the modulus of z and $\theta = \arctan(y/x)$, the argument of z.

Multiplication and division in polar form
$(r_1\,\text{cis}\,\theta_1)(r_2\,\text{cis}\,\theta_2) = r_1 r_2\,\text{cis}\,(\theta_1 + \theta_2)$
$(r_1\,\text{cis}\,\theta_1)/(r_2\,\text{cis}\,\theta_2) = (r_1/r_2)\,\text{cis}\,(\theta_1 - \theta_2)$
$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,(n\theta)$
$|z_1 z_2| = |z_1||z_2|$ $\quad |\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$

Complex conjugates
If $z = x + iy$, then $\overline{z} = x - iy$
$\overline{z \pm w} = \overline{z} \pm \overline{w}$ $\quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}$ $\quad \overline{\overline{z}} = z$
$z + \overline{z} = 2\,\text{Re}\,z$ $\quad z - \overline{z} = 2i\,\text{Im}\,z$ $\quad z\overline{z} = |z|^2$

Triangle inequality
$|z_1 + z_2| \leq |z_1| + |z_2|$

Roots of unity
Solutions of $z^n = 1$ are integer powers of ω, where $\omega = \text{cis}\,(2\pi/n)$.

Fundamental Theorem of Algebra
Every non-constant polynomial over the complex numbers has a (complex) root.
For polynomials with real coefficients, complex roots occur in conjugate pairs.

Transformation Geometry ($z \to w$)
Translations: $w = z + a$
Rotation about origin: $w = z\,\text{cis}\,\theta$
Reflection about x-axis: $w = \overline{z}$
Dilation about origin: $w = kz$ (k real)
All rigid transformations and spiral symmetries can be derived from these.

0.1 Problems

1. Suppose $ABCD$, $BEFC$ and $EGHF$ are 3 touching unit squares. Prove that $\angle BDC = \angle EDF + \angle GDH$.

2. Show that the equation of a general line in the complex plane is given by $\overline{a}z + a\overline{z} + b = 0$ $(a \in \mathbb{C}, b \in \mathbb{R})$

3. Using $\operatorname{cis}(n\theta) = (\operatorname{cis}\theta)^n$, derive formulae for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

4. Let ABC be a square inscribed in a circle and let P be any point on the circle and R the circumradius.

   (a) Show that $PA^2 + PB^2 + PC^2 + PD^2 = 8R^2$.

   (b) Locate all points such that the product $PA \cdot PB \cdot PC \cdot PD$ is a maximum or a minimum.

5. Let $A_1A_2 \dots A_n$ be a regular n-gon with circumradius 1. Show that $\prod_{i=2}^{n} |A_1A_i| = n$. Hence evaluate the product $\prod_{i=1}^{n-1} \sin\frac{i\pi}{n}$

6. A sequence of points $z_1, z_2, z_3, \dots$ is defined from an arbitary point z_0 by the transformations
$$z_n = c \cdot \overline{z_{n-1}} + c - 1$$
where $c = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Determine whether the sequence consists of repetitions of a finite set of points, and if so how many points there are in the set.

7. (Circle of Appolonius) Determine the locus of points $z \in \mathbb{C}$ such that $\frac{|z-a|}{|z-b|} = k$, where k is a positive real constant and a and b are fixed complex numbers.

8. Let ABC be a triangle with equilateral triangles BDC, CEA and AFB constructed externally on its sides. Prove that

   (a) $AD = BE = CF$

   (b) AD, BE and CF are concurrent.

9. (1999 Selection Exam) Let ABC be an arbitary triangle. Construct squares externally on sides AB and AC and let their centres be P and Q respectively. Let M be the midpoint of BC. Prove that PMQ is a right angled isosceles triangle.

10. Prove Ptolemy's inequality: In a quadrilateral $ABCD$, $AC \cdot BD \leq AB \cdot CD + BC \cdot DA$ with equality if and only if the quadrilateral is cyclic.

11. (IMO 1986/3) A triangle $A_1A_2A_3$ and a point P_0 are given in the plane We devine $A_s = A_{s-3}$ for $s \geq 4$. We construct a sequence of points $P_1, P_2, \dots$ such that P_{k+1} is the image of P_k under the clockwise rotation with centre A_{k+1} through 120 degrees (for $k = 0, 1, 2, \dots$). Prove that if $P_{1986} = P_0$ then the triangle $A_1A_2A_3$ is equilateral.