Diophantine Equations

1. Suppose that $ab = cd$. Show that there exist integers w,x,y and z such that $a = wx$, $b = yz$, $c = xy$, $d = zx$. This technique can be generalised to equations such as $ab = pcd$ for p prime. For this, there are two cases according to whether p divides a or b.

2. Find all positive integer solutions to the equation $x^2(x^2 + y) = y^{m+1}$.

3. (Engel) Do there exist integers m, n with $m^2 + (m + 1)^2 = n^4 + (n + 1)^4$?

4. (Engel) Find all integers k such that there exist positive integer solutions to the equation $x^2 + y^2 + z^2 = kxyz$.

5. Parametrise all three term arithmetic progressions of squares.

6. (Engel) If $ab = cd \neq 0$ then $a^2 + b^2 + c^2 + d^2$ is composite.

7. (1999 IMO Shortlist) Prove that there exists two strictly increasing sequences (a_n) and (b_n) such that $a_n(a_n + 1)$ divides $b_n^2 + 1$ for every natural number n.

8. (1999 Taiwan) Determine all triples (x, y, z) of positive integers such that
$$(x + 1)^{y+1} + 1 = (x + 2)^{z+1}.$$

9. (1999 Bulgaria) Prove that the equation
$$x^3 + y^3 + z^3 + t^3 = 1999$$
has infinitely many integer solutions.

10. (1995 Romanian training) Find all integer and positive solutions (x, y, z, t) of the equation
$$(x + y)(y + z)(z + x) = txyz$$
such that $(x, y) = (y, z) = (z, x) = 1$.

11. Find the smallest natural number n such that the sum of the squares of its divisors (including 1 and n) equals $(n + 3)^2$.

12. The positive integers a, b, x satisfy $x^{a+b} = a^b b$. Show that $a = x$ and $b = x^x$.

13. (1986 IMO Q1) Let d be any positive integer not equal to 2,5 or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

14. (1994 IMO Q4) Determine all ordered pairs (m, n) of positive integers such that
$$\frac{n^3 + 1}{mn - 1}$$
is an integer.

15. (1997 IMO Q5) Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

16. (1998 IMO Q4) Determine all pairs (a, b) of positive integers such that ab^2+b+7 divides $a^2b + a + b$.

17. (2001 IMO Q6) Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

18. Let p be a prime. Suppose that there exist integers a, b, x, y such that $p = a^2 + b^2 = x^2 + y^2$. Show that $\{a, b\} = \{x, y\}$.

19. Show that the following equation has infinitely many solutions in natural numbers:

$$x^3 + y^3 = z^4 - t^2$$