Inversion (and Transformation Geometry)

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1 Problems

1. Two common tangents of two intersecting circles meet at a point A. Let B be a point of intersection of the two circles, and C and D be the points in which one of the tangents touches the circles. Prove that the line AB is tangent to the circle passing through B, C and D.

2. Suppose that Γ is a cirle, internally tangent to two other intersecting circles Γ_1 and Γ_2 at points X and Y respectively. Γ_1 and Γ_2 intersect at A and B. C is a point of intersection of AB and Γ, and let the lines CX and CY meet Γ_1 and Γ_2 respectively at D and E (different from X and Y). Prove that DE is tangent to both Γ_1 and Γ_2.

3. Let ABC be a triangle, right angled at A, D the foot of the altitude from A, and E, F the incentres of triangles ABD and ACD respectively. EF intersects AB and AC in K and L. Show $AK = AL$.

4. Let $ABCD$ be a convex quadrilateral. On the segments AB, BC, CD, DA are chosen points M, N, P, Q respectively, such that

$$AQ = DP = CN = BM.$$

   Show that if $MNPQ$ is a square, then $ABCD$ is a square.

5. Prove Feuerbach's theorem; that the nine point circle is tangent to the incircle and the three excircles.

6. In the plane we are given two circles intersecting at X and Y. Prove that there exist four points with the following property: For every circle touching the given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, BD passes through one of these four points.

7. Let C be a circle inside another circle Γ. Let T_0 be a circle tangent externally to C and internally to Γ. For each $k \geq 1$, let T_k be the circle, distinct from T_{k-2} that is externally tangent to C and T_{k-1} and internally tangent to Γ. Prove that if $T_n = T_0$ for one particular such circle T_0, then $T_n = T_0$ for all possible positions of T_0.

8. Let A_1 be the centre of the square inscribed in an acute triangle ABC with two vertices on BC and one vertex each on AB and CA. Similarly define B_1 and C_1. Prove that the lines AA_1, BB_1 and CC_1 are concurrent.

9. (APMO 1998 Q4) Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line passing through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF respectively. Prove that AN is perpendicular to NM.

10. Let ABC be an acute triangle with circumcentre O and circumradius R. Let AO meet the circumcircle of OBC again at D, BO meet the circumcircle of OCA again at E, CO meet the circumcircle of OAB again at F. Show that $OD \cdot OE \cdot OF \geq 8R^3$.

11. Circle Γ_1 is internally tangent to circle Γ_2 at D. At a point B on Γ_1, different from D, the tangent to Γ_1 is drawn and intersects Γ_2 at A and C. Prove that BD bisects $\angle ADC$.

12. Let ABC be a triangle, let Γ be its incircle and Γ_a, Γ_b and Γ_c be three circles orthogonal to Γ passing through (B, C), (A, C) and (A, B), respectively. The circles Γ_a and Γ_b meet again in C'; and in the same way we obtain the points B' and A'. Prove that the radius of the circumcircle of $A'B'C'$ is half the radius of Γ.

13. $ABCD$ is a quadrilateral inscribed in the circle with diameter AD. Let O be the midpoint of AD, M the intersection of BC and AD, and K the second intersection point of the circumcircles of triangles BAO and CDO. Prove that $\angle MKO = 90°$.

14. Two circles intersect in points A, B. A line ℓ that contains the point A intersects the circles again in the points C, D. Let M, N be the midpoints of the arcs BC and BD respectively, which do not contain the point A, and let K be the midpoint of the segment CD. Show that $\angle MKN = 90°$.

15. (1990 USAMO Q5) An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

16. Let AD, BE, CF be altitudes of an acute triangle ABC with $AB > AC$. Line EF meets BC at P, and the line through D parallel to EF meets AC and AB at Q and R respectively. Let N be any point on the side BC such that $\angle NQP + \angle NRP < 180°$. Prove that $BN > CN$.