Number Theory (Senior)

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Theory

1. **Factorisations.** $x^n - 1$ and $x^m + 1$ for odd m both factorise. You should know this.

2. **Fermat's Little Theorem.** If p is a prime and a is an integer then
$$a^p \equiv a \pmod{p}.$$

3. **Euler's generalisation.** Suppose n is a positive integer and a is an integer coprime to n. Then
$$a^{\phi(n)} \equiv a \pmod{n}$$
where $\phi(n)$ is the Euler-phi function.

4. **Definition.** Suppose n is a positive integer and a is coprime to n. Then the *order* of a modulo n is equal to the smallest positive integer d such that
$$a^d \equiv 1 \pmod{n}.$$

5. **Lemma.** Let d be the order of a modulo n and suppose that $a^m \equiv 1 \pmod{n}$. Then d divides m.

Problems

1. Find all positive integers n for which n divides $2^n - 1$.

2. For which positive integers n is $2^{n-1} + 1$ divisible by n?

3. Show that
$$(a^m - 1, a^n - 1) = a^{(m,n)} - 1.$$

4. What is the largest power of 2 that divides $3^{2^n} - 1$?

5. (IMO 1999 Q4) Find all pairs (n, p) of positive integers such that
    - p is prime;
    - $n \leq 2p$;
    - $(p-1)^n + 1$ is divisible by n^{p-1}.

6. (Romania 1978) Show that for every natural number $a \geq 3$ there is an infinity of natural numbers n such that $n \mid a^n - 1$.

7. Let $m = (4^p - 1)/3$ where p is a prime and $p > 3$. Prove that 2^{m-1} has remainder 1 when divided by m.

8. Prove that for each positive integer n, there exist n consecutive positive integers none of which is an integral power of a prime number.

9. (IMO 1998 Q3) For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n) = k$ for some n.

10. (IMO 2000 Q5) Determine if there exists a positive integer n such that n has exactly 2000 prime divisors and $2^n + 1$ is divisible by n.

11. (IMO 1990 Q3) Determine all integers $n > 1$ such that
$$\frac{2^n + 1}{n^2}$$
is an integer.

12. (IMO 1998 Q3) For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n) = k$ for some n.

13. Let q be a positive rational number. Show that q can be expressed in the form
$$\frac{a^3 + b^3}{c^3 + d^3}$$
where a, b, c and d are positive integers.

14. Determine all positive integers n for which there exists an integer m so that $2^n - 1$ divides $m^2 + 9$.

15. Find all integers k such that there exist positive integers a and b satisfying
$$\frac{a+1}{b} + \frac{b+1}{a} = k.$$

16. The set of positive integers is partitioned into finitely many subsets. Show that some subset S has the following property: for every positive integer n, S contains infinitely many multiples of n.

17. Let p be a prime number greater than 5. Prove that the set $X = \{p - n^2 \mid n \in \mathbb{Z} \text{ and } n^2 < p\}$ contains two distinct elements x and y with $x \neq 1$ such that x divides y.

18. Let b, m, n be positive integers such that $b > 1$ and $m \neq n$. Prove that if $b^m - 1$ and $b^n - 1$ have the same prime divisors, then $b + 1$ is a power of 2.