BAILY-BOREL COMPACTIFICATION

PETER J MCNAMARA

1. Introduction

Let $G = Sp_{2g}$ be the symplectic group (Chevalley group scheme over $\mathbb{Z}$). Let K be a maximal compact subgroup of $G = G(\mathbb{R})$. The symmetric space $G(\mathbb{R})/K$ is commonly referred to as the Siegel upper half plane, we denote it by $\mathfrak{h}_g$. It is common to realise $\mathfrak{h}_g = \{A \in \mathrm{Mat}_g(\mathbb{C}) \mid {}^tA = A, \Im(A) > 0\}$. Thus it is a complex analytic space.

Let $\Gamma = G(\mathbb{Z})$ (respectively a congruence subgroup of). Then it is well known (what is the best reference?) that $\Gamma\backslash\mathfrak{h}_g$ parametrises principally polarised abelian varieties over $\mathbb{C}$ (respectively with level structure). If one wants to consider abelian varieties with a non-principal polarisation, then one works with a different $\mathbb{Z}$-form of Sp_{2g}.

Our aim is to construct a compactification X^B of $X = \Gamma\backslash\mathfrak{h}_g$ and show that it is a projective complex variety. This is the content of the paper of Baily and Borel. Let us make a few remarks about the generality of this work. Baily and Borel work with an arbitrary arithmetic quotient of a Hermitian symmetric domain. In this document, we will only work with quotients of the Siegel upper half space by a torsion-free Γ (and one can always make Γ torsion-free if one is willing to adopt the cost of passing to a finite index subgroup). This makes the exposition technically simpler, and should not obscure any of the key ideas in the proof. Unfortunately there is still a swathe of notation to wade through.

Theorem 1.1. *X^B is projective.*

Proof. Let $\mathcal{L}$ be some power of the canonical line bundle on X which extends to X^B via Theorem 6.3. We use a power of $\mathcal{L}$ to embed X^B into projective space. By Theorems 8.2 and 4.1, we can find sections of a sufficiently high power of $\mathcal{L}$ to separate points and tangent vectors (at least the latter for points of X).

Now consider some sections $E_0, \dots, E_n$ of $\mathcal{L}^k$ and the corresponding rational map from X^B to $\mathbb{P}^N$. The map is a closed immersion away from a closed analytic set. Pick a point p in the closed analytic set. Then by our results on separating points and tangent vectors, we can find a section E of $\mathcal{L}^{kl}$ such that E together with all degree l monomials of the E_i give a rational map from X^B to $\mathbb{P}^{N'}$ which is a closed immersion in a neighbourhood of p. Taking all monomials is the Segre embedding, so this latter map is well-behaved whenever the former map is, hence this new map is a closed immersion away from a strictly smaller analytic set.

Any decreasing chain of analytic sets in a compact space stabilises. □

Corollary 1.2. *The Baily-Borel compactification X^B is algebraic over $\mathbb{C}$.*

Proof. GAGA. □

2. The symplectic group and its subgroups

Let $\mathbf{G} = \mathbf{Sp_{2g}}$, defined as invertible matrices X with ${}^tXJX = J$ for $J = \begin{pmatrix} 0 & J_g \\ -J_g & 0 \end{pmatrix}$, with J_g the matrix with ones along the anti-diagonal and zeroes elsewhere. This means that the subgroup of upper-triangular elements of G constitutes a Borel subgroup. Let S be the torus, and $A = S(\mathbb{R})^0$.

Date: May 10, 2011.

For each b, we let P_b be the standard maximal parabolic subgroup with Levi factor $GL_b \times Sp_{2g-2b}$. Let $L_b = Sp_{2g-2b}$, Z_b be the kernel of the map from P_b to L_b and U_b be the unipotent radical of P_b.

The Cayley transform is given by conjugation by the matrix $\begin{pmatrix} I_g & iJ_g \\ iJ_g & I_g \end{pmatrix}$. Now consider the alternative inclusion of G in GL_{2g} by applying the Cayley transform. In this realisation, let $K_{\mathbb{C}}$ be the Levi of the Siegel parabolic, P^+ be the unipotent radical of the Siegel parabolic and P^- be the opposite unipotent subgroup. The notation for the Levi is chosen because it is the complexification of a maximal compact subgroup K of $G(\mathbb{R})$. Let $\mathfrak{p}^+$ be the (complex) Lie algebra of P^+. and use exp and log to denote the exponential map from $\mathfrak{p}^+$ to P^+ and its inverse.

For $g \in P^+K_{\mathbb{C}}P^-$, we denote by g_+ the part belonging to P^+ and g_0 the part belonging to $K_{\mathbb{C}}$.

The partial Cayley transform is conjugation by the matrix

$$c_b = \begin{pmatrix} I_b & 0 & iJ_b \\ 0 & I_{2g-2b} & 0 \\ iJ_g & 0 & I_g \end{pmatrix}.$$

3. The bounded realisation

Since $K_{\mathbb{C}}P^- \cap G = K$, there is an injection $\mathfrak{h}_g \to G_{\mathbb{C}}/(K_{\mathbb{C}}P^-)$ (the Lagrangian Grassmannian). There is also an open embedding $\mathfrak{p}^+ \to G_{\mathbb{C}}/(K_{\mathbb{C}}P^-)$ given by the exponential map $\mathfrak{p}^+ \to P^+$.

Lemma 3.1. *The image of* $\mathfrak{h}_g$ *is a bounded domain in* $\mathfrak{p}^+$.

Proof. Use the Cartan decomposition $G = KAK$. Note that K acts on $\mathfrak{p}^+$ in a linear fashion. Thus it suffices to check that the A-orbit of $0 \in \mathfrak{p}^+$ is bounded. This is a simple (essentially rank one) calculation. □

We call this the canonical bounded realisation of $\mathfrak{h}_g$.

The space $\Gamma\backslash\mathfrak{h}_g$ has a canonical line bundle $\mathcal{L}$ given by taking the top exterior power of the tangent bundle. Let first for motivational purposes ignore the issue of compactification and discuss the construction of the necessary sections of a power of $\mathcal{L}$.

Alt: the line bundle on $\mathfrak{h}_g$ is Γ-equivariant, so descends to the quotient.

If we pullback $\mathcal{L}$ to $\mathfrak{h}_g$, then any realisation of $\mathfrak{h}_g$ as a domain in $\mathbb{C}^N$ induces a trivialisation of $\pi^*(\mathcal{L})$. Sections of $\mathcal{L}^{\otimes l}$ can now be identified with holomorphic functions on $\mathfrak{h}_g$ satisfying $f(\gamma z) = J_\gamma(z)^l f(z)$, where $J_g(z)$ is the determinant of the Jacobian of the action of $g \in G$ at the point $z \in \mathfrak{h}_g$. For now we use the canonical bounded realisation of $\mathfrak{h}_g$.

Lemma 3.2.

$$\sum_{\gamma \in \Gamma} J_\gamma(z)^2$$

converges absolutely and uniformly on compact subsets of $\mathfrak{h}_g$.

Proof. Bound the sum (up to a constant) by the volume of $\mathfrak{h}_g$ in its bounded realisation. □

4. Poincaire series

Let f be a polynomial on $\mathfrak{p}^+$ and consider the bounded realisation $\mathfrak{h}_g \subset \mathfrak{p}^+$. Define the Poincaire series $P_f : \mathfrak{h}_g \longrightarrow \mathbb{C}$ by

$$P_f(z) = \sum_{\gamma \in \Gamma} J_\gamma(z)^l f(\gamma z).$$

The chain rule shows that this function satisfies the modularity condition once convergence issues are settled. Such convergence issues are covered by Lemma 3.2, and stronger results are discussed in Section 7

Theorem 4.1. *Suppose Γ is torsion free. Let $a_1, \dots, a_n$ be Γ-inequivalent points in $\mathfrak{h}_g$. Let $b_1, \dots, b_n$ be n complex numbers. Then for sufficiently large k, there exists a polynomial f in the bounded realisation of $\mathfrak{h}_g$ such that $J_f(a_i) = b_i$.*

Remark 4.2. The assumption Γ is torsion free is unnecessary, at the cost of insisting that k is divisible by some fixed integer. One can also obtain a more general result where b_i are replaced by prescribed p-th order Taylor series expansions, see [2] for more information.

Proof. Pick $0 < u < 1$. By Lemma 3.2, the set $\Gamma' = \{\gamma \in \Gamma \mid J_\gamma(a_i) < u\}$ is cofinite in Γ. Let f be a polynomial on $\mathfrak{p}^+$ with $f(a_i) = b_i$ and $f(\gamma a_i) = 0$ if $\gamma_i \notin \Gamma' \cup \{1\}$. Then $\lim_{l\to\infty} P_f(a_i) = b_i$. Now the map $f \mapsto (P_f(a_1), \dots, P_f(a_n))$ is linear. Hence the image is a subspace of $\mathbb{C}^n$. By our limit result, thus the image is surjective for large enough k. □

Granted this, we can separate points and tangent vectors in picking sections of line bundles. Unfortunately we do not have an appropriate Noetherianness property. We can only conclude that a decreasing sequence of closed analytic subsets of a complex analytic space is stationary if we know that our space is compact. Thus, we need to study what happens at infinity. In fact it will turn out that these sections constructed so far are all cuspidal.

5. Unbounded realisations

First we want to understand the closure of a Siegel set in the bounded realisation of $\mathfrak{h}_g$.

So suppose we have a sequence in $\mathfrak{S}$ converging in $\mathfrak{p}^+$. Thus we have sequences u_n in ω, a_n in A_t with $\lim_{n\to\infty} u_n a_n *$ existing. By compactness of ω, we can find a subsequence of u_n converging to $u \in \omega$. So $\lim_{n\to\infty} a_n *$ exists. (Use 1.9. The map $\omega \times \overline{\mathfrak{h}}_g \to K_{\mathbb{C}}$ has compact image). This implies that there exists b with $\lim a_n^{\alpha_i} = \infty$ for $i \leq b$ and is finite if $i > b$. Here $\alpha_1, \dots, \alpha_g$ are the simple roots with α_g long.

Thus the limit point is of the form $ua*_b$ for some $u \in \omega$ and $a \in A \cap L_b$. The action of Z_b on $*_b$ is trivial. So the closure of the Siegel set lies in $L_b *_b$, which is a hermitian symmetric space for a smaller symplectic group. In fact it is a Siegel set in L_b.

We define the standard boundary component to be $D_b = L_b *_b$. The group P_b acts on this component through the quotient L_b and the stabaliser of the point $*_b$ is a maximal compact subgroup of L_b.

We now define the standard unbounded realisation of $\mathfrak{h}_g$. Consider the partial Cayley transform is given by conjugation by $c_b \in G_{\mathbb{C}}$.

which is the usual Cayley transformation at the first b roots and the identity at the last $g - b$ roots (and thus acts trivially on L_b for example). This gives us a new embedding of $G_{\mathbb{R}}$ in $G_{\mathbb{C}}$. Consider the image of c_b in $G_{\mathbb{C}}/P^- K_{\mathbb{C}}$. The claim is that the G-orbit of this lies in $\mathfrak{p}^+$.

Lemma 5.1. *The image of $c_b G$ in the Lagrangian Grassmannian lies in $\mathfrak{p}^+$*

Proof. Think of this as the G-orbit of the point $\log(c_b)_+ \in \mathfrak{p}^+$ where G is acting on the Lagrangian Grassmannian via a c_b-conjugated action. Write $g \in G$ as qlk where $q \in Z_b$, $l \in L_b$ and $k \in K$. k acts trivially on the point $\log(c_b)_+$. The l-action is covered by the original Cartan decomposition argument in the bounded domain. Now $Z_b^{c_b} \subset K_{\mathbb{C}} P^+$ so preserves $\mathfrak{p}^+$. □

This yields another realisation of $\mathfrak{h}_g$ as a domain in $\mathfrak{p}^+$. Specifically $\mathfrak{h}_g$ is identified with the image of $c_b G$ and the G action is conjugated by c_b. We denote this (unbounded) realisation by S_b. The extreme cases are when $b = 0$, we recover the canonical bounded realisation while when $b = g$, we recover the classical realisation of $\mathfrak{h}_g$ mentioned in the introduction.

Given $g \in G$ and $X \in S_b$, we can explicate this action and write

$$g \cdot X = \log(g^{c_b} \exp(X))_+.$$

Let $\mathfrak{p}_b^+$ be the version of $\mathfrak{p}^+$ for L_b and D_b be the corresponding canonical bounded realisation of $\mathfrak{h}_b$. There is a natural quotient $\mathfrak{p}^+ \to \mathfrak{p}_b^+$ which induces a P_b-equivariant $\sigma_b : S_b \longrightarrow D_b$.

We write $J_\gamma^b(z)$ for the Jacobian in S_b, and $j_\gamma^b(z)$ for the Jacobian in D_b.

6. The Jacobian and extension of the line bundle

Proposition 6.1 (1.11(ii))**.** *If $g \in L_b$, we have $J_g^b(z) = j_g^b(\sigma_b(z))^q$ for some positive rational number q.*

Proof. Let $l \in L_b$ be such that $\sigma_b(z) = l*_b$. First, since $g \in P_b$, $J_g^b(-)$ is constant along the fibres of $\sigma_b : S_b \longrightarrow D_b$, we have $J_g^b(z) = J_g^b(l*_b) = J_{gl}^b(*_b)J_l^b(*_b)^{-1}$. By a similar use of the cocycle formula for j^b, it suffices to consider the case $z = *_b$.

Now we recall the formula [1, Lemma 1.9]

$$J_g^b(z) = \det \operatorname{Ad}_{\mathfrak{p}^+}((g^{c_b}\exp(z))_0).$$

L_b commutes with c_b, and also commutes with $\exp(*_b)$, which is in P^+. Thus for $g \in L_b$, we have $(g^{c_b}\exp(z))_0 = g_0$. Let $k = g_0 \in K_{\mathbb{C}} \cap L_b$. This reduces our proposition to the claim

$$\det \operatorname{Ad}_{\mathfrak{p}^+}(k) = \det \operatorname{Ad}_{\mathfrak{p}_b^+}(k)^q$$

for any $k \in K_{\mathbb{C}} \cap L_b$. Even without computation, it is obvious that such a q exists, as both sides are characters of a reductive group with rank one centre. □

Proposition 6.2. *If $p \in P_b$, then there are rational numbers n and q for which*

$$|J_p^b(z)| = |\chi(p)|^n |j_p^b(\sigma_b(z))|^q.$$

Here χ is a rational character of P_b.

Proof. Write $p = ml$ with $m \in Z_b$ and $l \in L_b$. Then $J_p^b(z) = J_m^b(lz)J_l^b(z)$. The first term is a function of m only, and hence by the cocyle relation a homomorphism from Z_b to $\mathbb{C}^\times$. Thus its absolute value is a power of $|\chi|$. We may do a calculation to determine the sign of n (which may be relevant later on).

For the second factor, we use the previous proposition to write it as $j_l^b(\sigma_b(z))^q$ which is equal to $j_p^b(\sigma_b(z))^q$ since m acts trivially on the boundary component. □

Theorem 6.3. *If k is an integer such that $kq \in \mathbb{Z}$ for all possible rational numbers q appearing in the above propositions, then $\mathcal{L}^k$ extends to a line bundle on X^B.*

From now on we will always assume that $kq \in \mathbb{Z}$ and $kn \in \mathbb{Z}$ for all rational numbers q and n appearing in the above propositions.

7. Poincaire-Eisenstein series

Write σ_b for the projection $\mathfrak{p}^+ \to \mathfrak{p}_b^+$ and let f be a polynomial on $\mathfrak{p}_b^+$. Define the Poincaire-Eisenstein series

$$E(z) = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} f(\sigma_b(\gamma z))J_\gamma^b(z)^k.$$

Here $\Gamma_0 = \Gamma \cap U_b$. One must check this is well defined.

Letting $\Gamma_\infty = \Gamma \cap P_b$, we introduce the function

$$Q(h) = \sum_{\lambda \in \Gamma_0 \backslash \Gamma_\infty} f(\sigma_b(\lambda h *_b))J_{\lambda h}^b(*_b)^k.$$

We expand our sum as

$$E(gK) = J_g^b(*_b)^{-k} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} Q(\gamma g).$$

The term in this sum corresponding to $\gamma = 1$ is the Poincaire series already known.

Lemma 7.1. *Let $p = ql$ and $\sigma_b(g*_b) = l_g*$. Here $q \in Z_b$, $l, l_g \in L_b$ and $g \in G$. We have*

$$Q(pg) = \chi(p)^{nk} J_g^b(*_b)^k j_{l_g}^b(*_b)^{-qk} \sum_{\lambda \in L_b \cap \Gamma} f(\sigma_b(\lambda l l_g *_b)) j_{\lambda l l_g}^b(*_b)^{qk}.$$

Proof. First note that $\lambda q \lambda^{-1} \in Z_b$ so acts trivially. Then we use the cocycle relation and Proposition 6.2 to end up at the desired result. □

Let $R_f(h) = f(\sigma_b(h*_b)) j_h^b(*)^{qk}$. This is a function on L_b.

Lemma 7.2. *$R_f(h)$ is left K_b-finite, $\mathfrak{z}$-finite and is in $L^1(L_b)$ if $qk \geq 2$.*

Proof. WLOG, just replace L_b by G. K acts linearly on $\mathfrak{p}^+$ through the adjoint action, so preserves the finite dimensional space $\mathrm{Sym}^n(\mathfrak{p}_+^*)$ which contains f. Now the left K translates of R_f are all of the form $R_{\mathrm{Ad}(k)f}$ so we have K-finiteness.

For the L^1-ness, WLOG $f = 1$ and we use the Cartan decomposition to reduce to showing that $\int_A J_a(*)da$ is finite. This is essentially a rank one computation.

Finally we note that the $\mathfrak{z}$-finiteness is automatic from holomorphicity and K-finiteness, as the following paragraph shows.

Note that for F to be holomorphic is equivalent to being annihilated by $\mathfrak{p}^-$. We use the PBW decomposition $U(\mathfrak{g}) = U(\mathfrak{p}^+) \otimes U(\mathfrak{k}) \otimes U(p^-)$. Pick an ordered basis of $\mathfrak{p}^+$ and $\mathfrak{p}^-$, and write $z \in \mathfrak{z}$ as $z = \sum_{m,n} X_m K_{mn} Y_n$ where X_k (resp. Y_k) is a sum of monomials of degree k in the basis of $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$), and $K_{mn} \in \mathfrak{k}$. Since z commutes with the centre of $\mathfrak{k}$, we have that only terms with $m = n$ appear in this sum. So $ZF = K_{00}F$ and from K-finiteness of F, we deduce $\mathfrak{z}$-finiteness. □

Proposition 7.3. *The sum $\sum_\lambda R(\lambda h)$ is uniformly absolutely convergent on compact sets, and is a bounded function (of h) on L_b.*

Proof. It is a theorem of Harish-Chandra [1, Theorem 5.4] that this is immediate from the above lemma. □

Proposition 7.4. *The Poincaire-Eisenstein series $E(z)$ defined above converges absolutely uniformly on compact sets.*

Proof. This follows from Theorem B.1 and the above estimates. □

8. Behaviour of P-E series in a Siegel Set

Consider the irreducible algebraic representation of G with highest weight χ^{nk}. Let v_h be highest weight vector and let $c(g) = ||\pi(g^{-1})v_h||^{-1}$. We compare this to the function Q from the previous section.

Proposition 8.1. *$Q(g) \leq Cc(g)$ for some constant C.*

Proof.

$$\frac{Q(pk)}{c(pk)} = \frac{Q(qlk)}{\chi^{nk}(p)c(k)} = \frac{\chi^{nk}(q)Q(lk)}{\chi^{nk}(p)} ||\pi(k^{-1})v_h||$$

is a product of bounded terms (the χ's cancel). □

We study our Poincaire-Eisenstein series in a Siegel set. (remark: we need to work with a translate of a Siegel set to cover all boundary components (Exercise!, or read [1])). So Let $u \in \omega$, $a \in A_t$, $k \in K$ and $\gamma \in \Gamma$. For $g = uak$, we have the estimate

$$J_g^b(*_b)^{-k} Q(\gamma g) \leq C' J_a^b(*_b)^{-k} c(\gamma u a k).$$

Note we have an explicit formula for the first factor.

The action of γ^{-1} on v_h turns it into $\sum_\mu v_\mu$ a sum of weight vectors. We will care which minimal weights appear. Action by u^{-1} doesn't change the set of minimal weights. a^{-1} acts on v_μ by multiplication by $a^{-\mu}$. As K is compact, $\pi(k)$ has operator norm bounded from above and below.

Putting this together, lets suppose that we travel along our Siegel set into a b'-boundary component. (yes I know there are less comparable boundary components).

If $b' \leq b$, for this term to survive (ie not converge to zero) in the limit on this boundary component, we must have that the action of γ can only subtract the weights of G that are also wieghts of L_c. Thus γ must lie in L_c. The P-E series in this case extends to what is a P-E series on the boudnary component. Note if $c = b$ we get our original Poincaire series (so can apply the separation results of Cartan).

If $b' > b$, we always converge to zero, ie our series is cuspidal. We end up with.

Suppose k is such that $nk \in \mathbb{Z}$ and $qk \in \mathbb{Z}$ for all n and q, and that k is sufficiently large.

Theorem 8.2. *The Poincaire-Eisenstein series $E_f(z)$ extends to an element of $\Gamma(X^B, \mathcal{L}^k)$. Its restriction to the b-boundary component is equal to a Poincaire series, and its restriction to any boundary component not containing this b-boundary component is zero.*

9. The topological structure

First the construction of the compactification as a set.

Define a rational boundary component to be a $G(\mathbb{Q})$-orbit of a standard boundary component. As a set, we take X^B to be the union of $\mathfrak{h}_g$ and all its rational boundary components (call this $\mathfrak{h}_g^*$), modulo the left action of Γ.

The topology (see [1, §4.8]) is defined by taking a fundamental system of neighbourhoods of some $x \in X^B$ to be all Γ_x-invariant subsets $x \in U \subset \mathfrak{h}_g^*$ such that the intersection of γU with a closure of a Siegel set containing γx is open in the closed Siegel set, for all γ. This turns X^B into a compact Hausdorff space with X an open dense subset.

10. The analytic structure

The main theorem which gives us an analytic structure is the following

V is a second countable Hausdorff space, a disjoint union of finitely many subspaces V_i, each of which is an irreducible normal analytic space.

On V one defines a sheaf of $\mathfrak{A}$-functions. An $\mathfrak{A}$-function on an open $U \subset V$ is a continuous complex-valued function on U whose restriction to each $U \cap V_i$ is analytic.

Theorem 10.1. [1, Theorem 9.2] *Suppose that*

1. *For each integer d, the union of all strata of dimension at most d is closed, and furthermore there is a unique stratum V_0 of maximal dimension which is open and dense in V.*
2. *Each point of V has a fundamental system of open neighbourhoods U_j with $U_j \cap V_0$ connected.*
3. *The restrictions to V_i of the sheaf of $\mathfrak{A}$-functions define the structure sheaf of V_i.*
4. *Each point has a neighbourhood U whose points are separated by the $\mathfrak{A}$-functions defined on U.*

Then V with its sheaf of $\mathfrak{A}$-functions is an irreducible normal analytic space.

11. Borel's Theorem

Via the theory of the Kobayashi invariant pseudo-distance, one can prove the following (I don't know what happens if Γ has torsion and $g \geq 2$).

Theorem 11.1. [3] *Let D be the open unit disc in $\mathbb{C}$ and $D^\times$ be the punctured unit disc. Suppose Γ is torsion-free. Then any holomorphic map from $D^a \times (D^\times)^b$ to X extends to a holomorphic map from D^{a+b} to $\overline{X}$.*

Now suppose Y is a smooth quasi-projective algebraic variety over $\mathbb{C}$. It is well known (resolution of singularities) that we can find an open embedding of Y into a projective variety $\overline{Y}$ such that $\overline{Y} \setminus Y$ is a normal crossings divisor. So locally for the complex analytic topology the embedding $Y \hookrightarrow \overline{Y}$ is of the form $D^a \times (D^\times)^b \hookrightarrow D^{a+b}$.

Thus by the above theorem, any holomorphic $f : Y \longrightarrow X$ can be extended to a holomorphic map $\overline{f} : \overline{Y} \longrightarrow \overline{X}$. This is a holomorphic map between the analytifications of two projective complex algebraic varieties, thus is algebraic.

Corollary 11.2. *The complex algebraic structure on* $\Gamma\backslash\mathfrak{h}_g$ *is unique.*

Appendix A. Siegel Sets

Let ω be a compact subset of U. Fix t and let $A_t = \{a \in A | a^\beta > t \;\forall\; \beta \in \Delta\}$ A set of the form $\omega A_t K$ is called a Siegel set.

Theorem A.1. *If* ω *is large enough and* t *is small enough then we have* $G(\mathbb{Z})\mathfrak{S} = G$.

Proof. Left multiplication by elements of $U(\mathbb{Z})$ means that $G = \omega AK$ for large enough ω. We concentrate on the torus part. We have $X_*(S) \otimes \mathbb{R} \simeq A$ via the exponential map. Write $g = ue^\lambda k$.

Pick $\alpha \in \Delta$ with $\lambda(\alpha)$ less than some absolute negative constant. (If such a simple root doesn't exist, there is nothing to prove). Write $u = u' e_\alpha(x)$ with $|x| \leq 1/2$ (WLOG via left multiplication by $U(\mathbb{Z})$). Consider $s_\alpha g = (u')^s e^{s\lambda} e_{-\alpha}(x\alpha(e^\lambda))sk$. Let us perform an Iwasawa decomposition on $e_{-\alpha}(x\alpha(e^\lambda))$ in the corresponding root subgroup of G.

This is a SL_2 calculation. The result is that when we write $sg = u' e^{\lambda'} k'$ we have from the SL_2 calculation that λ' lies on the line segment between λ and $s\lambda$.

If we keep applying this process, we improve λ in each iteration (it is now closer to any given point in the interior of the dominant chamber). There are no accumulation points via the SL_2 calculation, so after multiplying by enough elements of $G(\mathbb{Z})$, eventually we will end up with an element lying in $\omega A_t K$ as required. □

Corollary A.2. *If* G *has no rational characters then* $G(\mathbb{Z})\backslash G(\mathbb{R})$ *has finite volume.*

Proof. A Siegel set has finite volume. □

Theorem A.3. [4, Theorem 4.8, p193] *For any* $x \in G(\mathbb{Q})$*, the set of* $\gamma \in \Gamma$ *with* $\gamma\mathfrak{S} \cap x\mathfrak{S} \neq \emptyset$ *is finite.*

Appendix B. Convergence of Eisenstein Series

Let P be a parabolic subgroup of G and let χ be the character of P

$$\chi(p) = \det(\operatorname{Ad}(p)|_{\mathfrak{u}}).$$

We think of χ as a character of A.

Theorem B.1. *Let* $s \in \mathrm{Hom}(A, \mathbb{C}^\times)$ *be such that* $\Re(s) > \chi$*. Let* $f : G \longrightarrow V$ *be a function satisfying* $f(\gamma g) = f(g)$ *for all* $\gamma \in \Gamma_\infty$ *and such that*

$$\sup_{p \in P} ||\chi(p)^s f(pg)||$$

is bounded whenever g *runs over a compact set. Then the Eisenstein series*

$$E(f, g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma g)$$

converges absolutely uniformly on compact sets.

Proof. In the decomposition $G = KMAU$, the A component is uniquely determined, we write $a : G \longrightarrow A$ for the function this determines. By the Iwasawa decomposition, we can find the bound $||f(g)|| \leq Ca(g)^{-s}$ for some absolute constant C (Without loss of generality, assume s is real). So it suffices to consider the particular Eisenstein series

$$E(s,g) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} a(\gamma g)^{-s}.$$

Let us fix a compact subset Z of G. Then there exists a compact neighbourhood C of the identity in G such that for any fixed $g \in X$, the sets $\gamma g C$ with $\gamma \in \Gamma$ are disjoint. We get the bound

$$E(s,g) \leq \int_{\Gamma_\infty \backslash \Gamma g C} a(h)^{-s} dh.$$

Consider a highest weight representation V of G, with v_λ a highest weight vector stabilised by P, and contained in a lattice L stabilised by Γ. Let us write $\gamma g c = uma(\gamma g c)k$ and consider the equation $\gamma^{-1} v_\lambda = a(\gamma g c)^\lambda (gck^{-1}) v_\lambda$. The term $||\gamma^{-1} v_\lambda||$ is bounded from below since it is the length of a vector in the lattice L, and $||gck^{-1} v_\lambda||$ is bounded from above for $g \in Z$, $c \in C$ and $k \in K$ are all running over compact sets. Thus we get $a(\gamma g c)^\lambda$ is bounded from above.

Varying λ, we find a constant t such that $a(\gamma g c) \in A_t^* = \{a \in A | a^\omega < t\}$ where one is running over all relevant positive fundamental weights ω.

We require the fact that there exists some set $\omega \in U.M$ of finite measure with $U.M = \Gamma_\infty \omega$ (Corollary A.2). Now we have

$$E(s,g) < \int_{\omega A_t^* K} a(h)^{-s} dh = \operatorname{vol}(\omega) \operatorname{vol}(K) \int_{A_t^*} a^{\chi - s} da.$$

This is a product of integrals of the form $\int_0^T y^{s-1} dy$ with $s > 0$, hence converges as required. □

References

1. Baily and Borel. Compactification of arithmetic quotients of bounded symmetric domains.
2. Blanchard and Cartan. Rectifications à l'exposé 1 du séminaire 1953/54. Séminaire Henri Cartan, 10 no. 1 (1957-1958), Exposé No. 10 bis, 6 p. http://www.numdam.org/numdam-bin/fitem?id=SHC_1957-1958__10_1_A12_0
3. Borel. Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem.
4. Platonov and Rapinchuk. Algebraic Groups and Number Theory.

E-mail address: petermc@math.stanford.edu