Important remarks about this paper

This paper relies on the construction of some autoequivalences of $K(\mathcal{U})$, the homotopy category of a categorified quantum group. Such autoequivalences are still conjectural. The paper [ALELR] is only able to produce functors from $\mathcal{U}$ to $K(\mathcal{U})$, which is not enough as taking the homotopy category is not an idempotent operation. The arguments here will be left frozen in time and no results claimed in this paper can be assumed to be true.

ON A BRAID GROUP ACTION

PETER J MCNAMARA

Abstract. We discuss some consequences of the braid group action on a categorified quantum group. Results include a description of reflection functors for quiver Hecke algebras and a theory of restricting categorical representations along a face.

1. Introduction

It is a classical construction to associate to any symmetrisable Cartan matrix a quantum group $U_q(\mathfrak{g})$. We concern ourselves with a categorified version of this construction, where a strict 2-category $\mathcal{U}$ is produced whose Grothendieck group is canonically identified with the idempotented form of the corresponding quantum group.

This theory of a categorical quantum group originates with the work of Chuang and Rouquier [CR08] who used the notion of a categorical $\mathfrak{sl}_2$ action to construct interesting derived equivalences. The generalisation to general $\mathfrak{g}$ came in the work of Khovanov and Lauda [KL10] and Rouquier [Rou12]. They give two different presentations of a 2-category $\mathcal{U}$, the equivalence of which was shown by Brundan [Bru].

The braid group acts by algebra automorphisms on the quantum group $U_q(\mathfrak{g})$. The starting point of this paper is the hypothesis that this braid group action can be lifted to a braid group action on the homotopy category $K(\mathcal{U})$ by autoequivalences $\mathcal{T}_s$. Results in this direction are obtained in [ALELR] where the appropriate functors from $\mathcal{U}$ to $K(\mathcal{U})$ are obtained, but the extension to an autoequivalence of $K(\mathcal{U})$ is still open. They work under the assumption that the Cartan matrix is simply laced. In this paper we explore the implications of the exitence of this action. The applications which we study all require the theory of standard modules for KLR algebras. This theory was developed in [BKM] in finite type and in [McNb] in affine type over a field of characteristic zero. The necessary facts from this theory are all recalled when they are needed. We also prove that these standard modules are compatible with these autoequivalences $\mathcal{T}_s$ in a precise way in Proposition 8.2.

The first application is to the construction of reflection functors for KLR algebras. These are functors which categorify the Satio reflection on the crystal $B(\infty)$, as well as Lusztig's braiding automorphism T_s, restricted to the positive part of the quantum group. These T_s

Date: October 25, 2018.

induce isomorphisms $\ker({}_s r) \cong \ker(r_s)$ inside $U_q(\mathfrak{g})^+$. Both subspaces $\ker({}_s r)$ and $\ker(r_s)$ are categorified by a Serre subcategory of the category of quiver Hecke modules, which we denote by ${}_s\mathcal{C}$ and $\mathcal{C}_s$ respectively. We show how the autoequivalence $\mathcal{T}_s$ induces an equivalence of the abelian categories ${}_s\mathcal{C}$ and $\mathcal{C}_s$.

This equivalence was obtained geometrically in finite simply laced type over a field of characteristic zero in [K1]. This was subsequently generalised to finite simply laced type in all characteristics in [McNa]. The related work of [XZ, Zha] provides a geometric incarnation of T_s in all types but does not produce results of the same strength that we provide here.

The second application is to a theory of restricting a categorical representation. In [MT], the notion of a face of a root system is introduced. Each face defines a root system and hence there is a KLR algebra associated to that face, which we will call R_F. The main result of [MT] is the construction of a fully faithful functor from R_F-mod to R-mod, whose essential image is explicitly determined in terms of a subcategory of cuspidal modules if the face root system is of finite type. In this paper we go beyond the results of [MT] and show that each face induces a functor between the homotopy categories of categorified quantum groups. This allows us to restrict a categorical action on a category $\mathcal{C}$ to a categorical action for the face quantum group on the homotopy category $K(\mathcal{C})$. In certain cases the restricted categorical action is actually an action on $\mathcal{C}$ and we give a criterion for checking this.

2. Notations

Given two objects X and Y, we write qX for the grading shift of X, $\mathrm{Hom}(X,Y)$ for the degree zero morphisms from X to Y and

$$\mathrm{HOM}(X,Y) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}(q^d X, Y)$$

for the graded vector space of all morphisms.

3. The 2-category

For us a 2-category will always be a strict 2-category, which is the same thing as a category enriched in categories, i.e. the homomorphisms between two objects forms a category. In fact, our 2-categories will usually be categories enriched in Karoubian triangulated categories.

Let S be a set and $A = (a_{st})_{s,t\in S}$ be a simply-laced Cartan matrix. This means that $a_{ss} = 2$ and if $s \neq t$, $a_{st} = a_{ts} \in \{0, -1\}$. Choose also a realisation of A. This is the additional data of a complex vector space $\mathfrak{h}$, a set of linearly independent vectors $\alpha_s \in \mathfrak{h}^*$ and a set of linearly independent vectors $\alpha_s^\vee \in \mathfrak{h}$ such that $\langle \alpha_s^\vee, \alpha_t\rangle = a_{st}$. Define the weight lattice

$$P = \{\lambda \in \mathfrak{h}^* \mid \langle \alpha_s^\vee, \lambda\rangle \in \mathbb{Z} \text{ for all } s \in S\}.$$

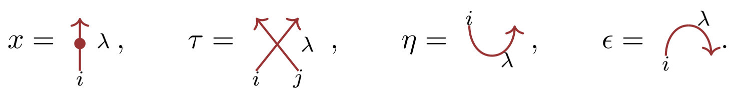
The reason for restricting to simply-laced type is that this is the generality under which the autoequivalences of [ALELR] are constructed. We expect that generalising the results of [ALELR] to all types will allow the results of this paper to similarly become available in greater generality, subject still to the requirements that the necessary theory of standard modules for the relevant KLR algebras exists.

For each ordered pair (i,j) of distinct elements of Π, let t_{ij} be a nonzero element of k. To this data, there is a Kac-Moody 2-category $\mathcal{U}$. It has

Objects $\lambda \in P$.
generating 1-morphisms:

$$\mathcal{E}_s 1_\lambda : \lambda \longrightarrow \lambda + \alpha_s, \qquad \mathcal{F}_s 1_\lambda : \lambda \longrightarrow \lambda - \alpha_s$$

generating 2-morphisms:

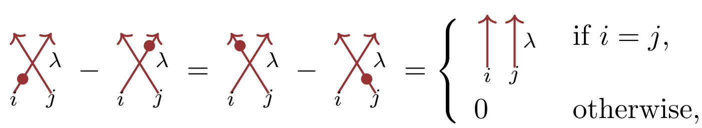


(3.1)

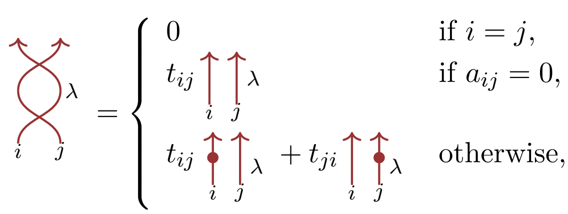
subject to a list of relations. By work of Brundan [Bru], different choices of relations that appear in the literature give equivalent 2-categories.

First, we have the isotopy relations. These state that two diagrams which are isotopic are equal.

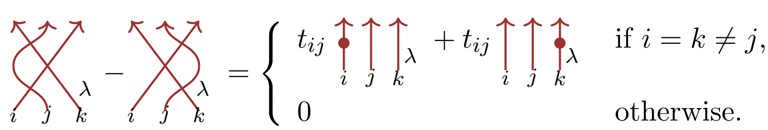
Second, we have the quiver Hecke relations:



(3.2)

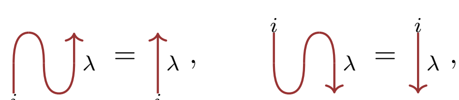


(3.3)

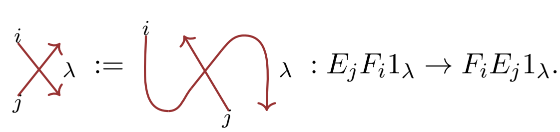


(3.4)

Third, we have the right adjunction relations

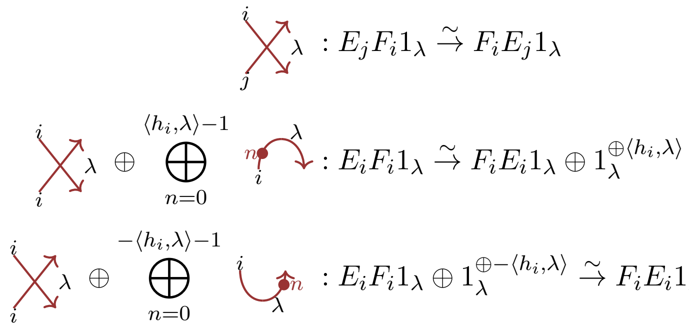

(3.5)

In order to describe the remaining relations, we first introduce a new 2-morphism.



(3.6)

Then we require that the following 2-morphisms are isomorphisms:



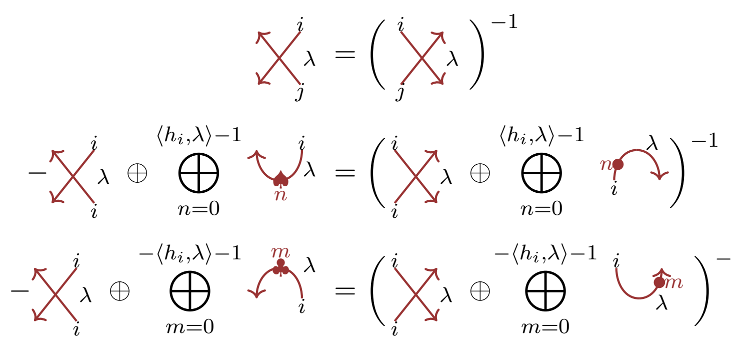
$$: E_jF_i1_\lambda \xrightarrow{\sim} F_iE_j1_\lambda \qquad \text{if } i \neq j, \tag{3.7}$$

$$\oplus \bigoplus_{n=0}^{\langle h_i,\lambda\rangle -1} : E_iF_i1_\lambda \xrightarrow{\sim} F_iE_i1_\lambda \oplus 1_\lambda^{\oplus\langle h_i,\lambda\rangle} \qquad \text{if } \langle h_i,\lambda\rangle \geq 0, \tag{3.8}$$

$$\oplus \bigoplus_{n=0}^{-\langle h_i,\lambda\rangle -1} : E_iF_i1_\lambda \oplus 1_\lambda^{\oplus-\langle h_i,\lambda\rangle} \xrightarrow{\sim} F_iE_i1_\lambda \qquad \text{if } \langle h_i,\lambda\rangle \leq 0. \tag{3.9}$$

where the label n next to a dot means that this strand carries n dots.

This means that there are some additional generators which are two-sided inverses to the 2-morphisms in (3.7)–(3.9), we draw them as  $\lambda : F_iE_j1_\lambda \to E_jF_i1_\lambda$,  $\lambda : 1_\lambda \to E_iF_i1_\lambda$ and  $\lambda : F_iE_i1_\lambda \to 1_\lambda$ for $0 \leq n < \langle h_i,\lambda\rangle$ and $0 \leq m < -\langle h_i,\lambda\rangle$ and they satisfy the following relations:



$$\text{if } i \neq j,$$

$$\text{if } \langle h_i,\lambda\rangle \geq 0,$$

$$\text{if } \langle h_i,\lambda\rangle \leq 0.$$

We now introduce some more 2-morphisms, the leftward cups and downward arrows, in order to describe Webser's nondegeneracy theorem.

Let $\mathcal{B}_\lambda$ be the commutative k-algebra freely generated by all bubbles.

[KL10, Definition 3.15] is a definition of nondegenerate.

Theorem 3.1 (Webster's nondegeneracy theorem)**.** [Web, Theorem A] *Basis of Hom spaces between products of $\mathcal{E}$s and $\mathcal{F}$s is as large as possible.*

Out of $\mathcal{U}$, for $* \in \{b,+,-,\emptyset\}$ we construct a new 2-category $K^*(\dot{\mathcal{U}})$. Like $\mathcal{U}$, its objects are elements of P. For any two elements $\lambda,\mu \in P$, the morphism category in $K^*(\dot{\mathcal{U}})$ is defined to be

$$\mathrm{Hom}_{K^*(\dot{\mathcal{U}})}(\lambda,\mu) = \mathrm{Kar}(K^*(\mathrm{Hom}_{\mathcal{U}}(\lambda,\mu)))$$

where Kar refers to taking the Karoubian envelope.

4. Summary of Necessary Results

We assume the existence of a triangulated autoequivalence $\mathcal{T}_s$ of $K^b(\dot{\mathcal{U}})$ for each $s \in S$ satisfying the following properties:

If s and t are connected by an edge, then

$$\mathcal{T}_s(\mathcal{F}_t 1_\lambda) = q\mathcal{F}_t\mathcal{F}_s 1_{s\lambda} \to \clubsuit \mathcal{F}_s\mathcal{F}_t 1_{s\lambda}, \tag{4.1}$$

$$\mathcal{T}_s(q\mathcal{F}_s\mathcal{F}_t 1_\lambda \to \clubsuit \mathcal{F}_t\mathcal{F}_s 1_\lambda) = \mathcal{F}_t 1_{s\lambda} \tag{4.2}$$

where the differential in each case is the downward crossing. The symbol $\clubsuit$ denotes homological degree zero.

Theorem 4.1. [ALELR] *$\mathcal{T}_s$ decategorifies to T_s.*

5. Quiver Hecke Algebras

The KLR algebra R is the algebra generated by the upward strands, subject to the isotopy and quiver Hecke relations.

For $\nu \in \mathbb{N}I$, let $R(\nu)$ be the subalgebra of R consisting of diagrams where the colours of the strands add to ν.

By placing strands next to each other there is a nonunital inclusion of algebras $R(\lambda) \otimes R(\mu) \hookrightarrow R(\lambda+\mu)$. There is thus a corresponding induction functor

$$\mathrm{Ind} : R(\lambda)\text{-mod} \times R(\mu)\text{-mod} \longrightarrow R(\lambda+\mu)\text{-mod}$$

written $(M, N) \mapsto M \circ N$, given by

$$M \circ N := R(\lambda+\mu)e \otimes_{R(\lambda)\otimes R(\mu)} (M \otimes N)$$

where e is the image of the unit under the algebra inclusion.

Let $P_{\mathbf{i}_1\cdots\mathbf{i}_n} = R(\mathbf{i})e(\mathbf{i}) = P_{\mathbf{i}_1} \circ \cdots \circ P_{\mathbf{i}_n}$. This is a projective $R(\mathbf{i})$-module.

There is the adjunction [McN15]

$$\mathrm{Ext}^i(A, B \circ C) \cong \mathrm{Ext}^i(\mathrm{Res}\, A, C \boxtimes B). \tag{5.1}$$

6. The embeddings of categories

For each $\lambda \in P$, there is a functor

$$K^-(R(\nu)\text{-pmod}) \xrightarrow{i_\lambda} \mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda, \lambda+\nu).$$

This functor sends the projective $R(\nu)$-module $P_{\mathbf{i}_1\cdots\mathbf{i}_n}$ to $\mathcal{F}_{\mathbf{i}_1}\cdots\mathcal{F}_{\mathbf{i}_n}1_\lambda$

These functors satisfy a compatibility between induction and composition.

$$\begin{array}{ccc} K^-(R(\mu)\text{-pmod}) \times K^-(R(\nu)\text{-pmod}) & \xrightarrow{\quad\mathrm{Ind}\quad} & K^-(R(\mu+\nu)\text{-pmod}) \\ \big\downarrow{\scriptstyle i_{\lambda-\nu}\times i_\lambda} & & \big\downarrow{\scriptstyle i_\lambda} \\ \mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda-\nu, \lambda-\mu-\nu) \times \mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda, \lambda-\nu) & \longrightarrow & \mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda, \lambda-\mu-\nu) \end{array}$$

where the horizontal map along the last row is the composition in $K^-(\dot{\mathcal{U}})$.

For each $\lambda \in P$, let $\mathcal{B}_\lambda$ be the endomorphism ring of the unit in the monoidal category $\mathrm{Hom}_{\mathcal{U}}(\lambda, \lambda)$. This space consists purely of bubbles. In degree zero, it is spanned by the identity and in negative degrees, it is zero.

Theorem 6.1. *Suppose X and Y are two objects in $K^-(R(\nu))$-pmod). Then there is an isomorphism of graded vector spaces.*

$$\mathrm{Hom}(i_\lambda(X), i_\lambda(Y)) \cong \mathrm{Hom}(X, Y) \otimes \mathcal{B}_\lambda.$$

Proof. Since X and Y are bounded above, we can find bounded above chain complexes $P_\bullet$ and $Q_\bullet$ representing X and Y respectively such that each module P_i and Q_i is a direct sum of standard projective modules $R(\nu)e_{\mathbf{i}}$.

Let $\{h_j\}$ be a basis of the space $\mathcal{B}_\lambda$.

Let $f_\bullet : i_\lambda(P_\bullet) \longrightarrow i_\lambda(Q_\bullet)$ be a morphism of chain complexes. We can express each f_i in the form $f_i = \sum_j g_{ij} \otimes h_j$ where each g_{ij} has no bubbles, i.e. comes from a morphism in $K^-(R(\nu)\text{-mod})$.

That $f_\bullet$ is a chain map is expressed in the identity

$$\sum_j dg_{ij} \otimes h_j = \sum_j g_{i+1,j} d \otimes h_j.$$

By Webster's nondegeneracy theorem, this implies that for each j, $dg_{ij} = g_{i+1,j}d$. Thus, the collection $g_{\bullet,j}$ is a chain map in $K^-(R(\nu)\text{-pmod})$.

This shows that the map from $\mathrm{Hom}(X, Y) \otimes \mathcal{B}_\lambda$ to $\mathrm{Hom}(i_\lambda(X), i_\lambda(Y))$ is surjective. A similar argument shows that if f is homotopic to zero, then this homotopy comes from homotopies between each g_{ij} and zero. Therefore this map is injective also. □

Corollary 6.2. *The functor i_λ is faithful.*

7. Standard modules

We summarise the current state of the theory of standard modules for quiver Hecke algebras. This theory is currently known to exist in finite type in all characteristics and in symmetric affine type when k is of characteristic zero. The references are [BKM] in the former case and [McNb] in the latter.

Let Φ^+ be the set of positive roots. A *convex order* on Φ^+ is a preorder $\prec$ such that

- If S and T are two subsets of Φ^+ such that $s \prec t$ for all $s \in S$ and $t \in T$ then
$$\mathrm{span}_{\mathbb{R}_{\geq 0}} S \cap \mathrm{span}_{\mathbb{R}_{\geq 0}} T = \{0\},$$
- If $s \preceq t$ and $t \preceq s$ then s and t are proportional.

Let $\alpha \in \Phi^+$. A representation M of $R(\alpha)$ is said to be semicuspidal (with respect to the convex order $\prec$) if $\mathrm{Res}_{\beta\gamma} M \neq 0$ implies that β is a sum of roots less than α and γ is a sum of roots greater than α.

Let α be an indivisible root. An indecomposable projective object in the category of semicuspidal $R(\alpha)$-modules is called a root module. The grading shift on these root modules is customarily normalised such that their heads are self-dual. For each indecomposable root α, the number of root modules for α is equal to the dimension of the root space $\mathfrak{g}_\alpha$. In particular, if α is a real root, there is a unique root module, which we call $\Delta(\alpha)$.

We consider the standard modules introduced in [BKM] and [McNb]. These depend on the convex order $\prec$ and are built out of root modules. The root modules corresponding to real roots have already been introduced, these are the modules $\Delta(\alpha)$. For the indivisible imaginary root δ, we will call the modules denoted $\Delta(\omega)$ in [McNb] root modules. These are the projective modules in the category of cuspidal $R(\delta)$-modules.

Standard modules are naturally indexed by root partitions. A root partition is a sequence $\lambda = (\alpha_1^{n_1}, \cdots, \alpha_l^{n_l})$ where $\alpha_1 \succ \cdots \succ \alpha_l$ are indivisible roots, each n_i is a positive integer unless $\alpha_i = \delta$, in which case it is a collection of partitions. To each term $\alpha_i^{n_i}$ a standard module $\Delta(\alpha_i)^{(n_i)}$ is constructed. If α_i is real then $\Delta(\alpha_i)^{\circ n_i}$ is a direct sum of $n_i!$ copies of the module $\Delta(\alpha_i)^{(n_i)}$ with grading shifts. If α_i is imaginary then $\Delta(\alpha_i)^{(n_i)}$ is a summand of a product of certain modules $\Delta(\omega)$ of weight δ in $\mathcal{C}_F$; see [McNb] for the details (where this module is denoted $\Delta(\underline{\lambda})$). The standard module is then defined to be the indecomposable module

$$\Delta(\lambda) = \Delta(\alpha_1)^{(n_1)} \circ \cdots \circ \Delta(\alpha_l)^{(n_l)}.$$

In [BKM] and [McNb] homological properties of these modules are developed which justify the use of the name standard.

If α is a real root, let $L(\alpha)$ be the head of $\Delta(\alpha)$ and let $L(\alpha^n) = L(\alpha)^{\circ n}$. If $\underline{\lambda}$ is a multipartition, let $L(\underline{\lambda})$ be the head of $D(\underline{\lambda})$. These modules are all simple.

Let $\lambda = (\alpha_1^{n_1}, \cdots, \alpha_l^{n_l})$ be a root partition. Define the proper costandard module

$$\overline{\nabla}(\lambda) = L(\alpha_l^{n_l}) \circ \cdots \circ L(\alpha_1^{n_1}).$$

Theorem 7.1. *The simple modules for $R(\nu)$ are classified up to isomorphism and grading shift by root partitions. The simple module $L(\lambda)$ corresponding to the root partition λ is the socle of $\overline{\nabla}(\lambda)$. Furthermore every other simple subquotient $L(\mu)$ of $\overline{\nabla}(\lambda)$ has $\lambda \prec \mu$ in the lexicographical order on root partitions.*

Proposition 7.2. [McNb, Proposition 24.3] *Let Δ be a standard module and $\overline{\nabla}$ be a proper standard module. Then for $i > 0$,*

$$\mathrm{Ext}^i(\Delta, \overline{\nabla}) = 0.$$

Lemma 7.3. *Let Δ be a root module. Then there are root modules Δ_β and Δ_γ and a nonzero q-integer m such that there is a short exact sequence*

$$0 \to q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma \xrightarrow{f_{\beta\gamma}} \Delta_\gamma \circ \Delta_\beta \to \Delta^{\oplus m} \to 0.$$

If $wt(\Delta)$ is not minimal in $\Phi^+ \setminus \{\alpha_s\}$, then Δ_β and Δ_γ can be chosen to be in $\Phi^+ \setminus \{\alpha_s\}$. Furthermore $f_{\beta\gamma}$ spans $\mathrm{Hom}(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta)$ and $\mathrm{HOM}(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta)$ is concentrated in nonnegative degrees.

Proof. In finite type, this is [BKM, Theorem 4.10]. In symmetric affine type, this is [McNb, Lemma 16.1] if $\mathrm{wt}(\Delta)$ is real and [McNb, Theorem 17.1] if $\mathrm{wt}(\Delta)$ is imaginary. The statement about $\mathrm{Hom}(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta)$ being one-dimensional is not explcitly mentioned in these references, but is clear from the proofs. Note that we can replace $\Phi^+\setminus\{\alpha_s\}$ by any interval. □

Proposition 7.4. *Every standard module is obtained from a root module by a process of induction and taking direct summands.*

Remark 7.5. We expect that if a theory of standard modules is developed in affine type over a field of positive characteristic, it will not satisfy Proposition 7.4.

Definition 7.6. *A module M is said to have a Δ-flag if there exists a filtration by submodules*

$$M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

such that each subquotient M_{i+1}/M_i is a standard module.

Lemma 7.7. [BKM, Theorem 3.13] *A finitely generated module M has a Δ-flag if and only if $\operatorname{Ext}^1(M, \overline{\nabla}) = 0$ for all proper costandard modules $\overline{\nabla}$.*

Pick a convex order ${}_s\prec$ which has α_s as its largest element. From ${}_s\prec$ we can obtain another convex order $\prec_s$, by

- $\alpha_s \prec_s \beta$ for all $\beta \in \Phi^+ \setminus \{\alpha_s\}$
- $\beta \prec_s \gamma$ if $s\beta \,{}_s\!\prec s\gamma$ for all $\beta, \gamma \in \Phi^+ \setminus \{\alpha_s\}$

These convex orders induce two families of standard modules in R-mod, we denote them by ${}_s\Delta(\lambda)$ and $\Delta_s(\lambda)$.

Lemma 7.8. *Suppose P is projective in ${}_s\mathcal{C}$ (or in $\mathcal{C}_s$). Then $\operatorname{Ext}^i_R(P, M) = 0$ for all R-modules M in ${}_s\mathcal{C}$ (respectively $\mathcal{C}_s$) and $i > 0$.*

Note that the Ext group is computed in the category of R-modules rather than in ${}_s\mathcal{C}$ (or $\mathcal{C}_s$), so this result is not a tautology for $i > 1$.

Proof. We consider the case where P is projective in ${}_s\mathcal{C}$, the other case following similarly. First we prove that P has a Δ-flag. By Lemma 7.7 it suffices to prove that $\operatorname{Ext}^1(P, \overline{\nabla}) = 0$ for all proper costandard modules $\overline{\nabla}$. If λ is a root partition with $\lambda(\alpha_s) = 0$, then $\overline{\nabla}(\lambda) \in {}_s\mathcal{C}$. So in this case $\operatorname{Ext}^1(P, \overline{\nabla}(\lambda)) = 0$ as P is projective in ${}_s\mathcal{C}$. If $\lambda(\alpha_s) \neq 0$, then $\overline{\nabla}(\lambda) \cong X \circ L_s$ for some X. Then by (5.1),

$$\operatorname{Ext}^1(P, \overline{\nabla}(\lambda)) \cong \operatorname{Ext}^1(\operatorname{Res}_{\alpha_s, \nu-\alpha_s} P, L_s \otimes X).$$

Since $P \in_s \mathcal{C}$, this restriction is zero and hence this Ext group is zero.

Therefore we have succeeded in showing that $\operatorname{Ext}^1(P, \overline{\nabla}) = 0$ for all proper costandard modules $\overline{\nabla}$, hence P has a Δ-flag. By Propsition 7.2, this implies that $\operatorname{Ext}^i(P, \overline{\nabla}) = 0$ for all proper costandard modules $\overline{\nabla}$ and $i > 0$.

We now turn our attention to the statement that $\operatorname{Ext}^i_R(P, M) = 0$ for all R-modules M in ${}_s\mathcal{C}$.

Without loss of generality we may assume M is simple. Then M injects into a proper costandard module $\overline{\nabla}$. Let Q be the quotient $\overline{\nabla}/M$. Then every simple subquotient L of Q satisfies $L \prec M$.

By induction on the partial preorder $\prec$, we may assume $\operatorname{Ext}^i(P, L) = 0$ for all $L \prec M$ and $i > 0$. Hence $\operatorname{Ext}^i(P, Q) = 0$ for $i > 0$. Now apply $\operatorname{Hom}(P, -)$ to the short exact sequence

$$0 \to M \to \overline{\nabla} \to Q \to 0.$$

The resulting long exact sequence implies that $\operatorname{Ext}^i(P, M) = 0$ for $i \geq 2$.

This leaves only the $i = 0$ case, but $\operatorname{Ext}^1(P, M)$ vanishes since P is projective in ${}_s\mathcal{C}$ and M is in ${}_s\mathcal{C}$. □

8. Reflection Functor

Let R be a quiver Hecke algebra and $s \in S$. Let ${}_se$ be the sum of all generating idempotents with first strand coloured s. Let e_s be the sum of all generating idempotents with last strand coloured s. Let ${}_s\mathcal{C}$ (respectively $\mathcal{C}_s$) be the full subcategory of R-modules on which ${}_se$ (respectively e_s) acts by zero.

Equivalently

$$ {}_s\mathcal{C} = R/\langle {}_se\rangle\text{-mod} \quad \text{and} \quad \mathcal{C}_s = R/\langle e_s\rangle\text{-mod}. $$

Fix $s \in S$. For each s, we have inclusions of full subcategories

$$ {}_s\mathcal{C}, \mathcal{C}_s \subset R\text{-mod} \subset K^-(R\text{-pmod}). $$

If our Cartan datum is of finite type then R has finite global dimension by [McN15, Theorem 4.7]. So in this case, the essential image lies in the bounded homotopy category $K^b(R\text{-pmod})$. In general, the standard modules have finite projective dimension so lie in $K^b(R\text{-pmod})$, while R in general has infinite global dimension.

Theorem 8.1. *Suppose our quiver Hecke algebra is simply laced, of finite or affine type. If we are in affine type, assume furthermore that the ground field is of characteristic zero. Then there is a monoidal equivalence of categories ${}_s\mathcal{C} \cong \mathcal{C}_s$ which decategorifies to Lusztig's braid group automorphism.*

The restriction to simply laced is due to the generality of [ALELR]. The further restrictions are needed since we rely on the theory of standard modules, developed in [BKM] (building on [McN15]) in finite type and in [McNb] in affine type over a field of characteristic zero.

In finite type ADE and in characteristic zero, Kato [K1] has proved this equivalence geometrically. In finite type ADE and in all characteristics there is a geometric proof in [McNa] The paper of Xiao and Zhao [XZ] provides an interpretation of T_s in terms of perverse sheaves in all symmetric types, again in characteristic zero, which is subsequently generalised to all symmetrisable types in [Zha]. Kato [K2] has shown how to use this to define $\mathcal{T}_s$ geometrically in characteristic zero in all symmetric types. A geometric approach to the monoidality of this functor, restricted again to characteristic zero, is in [McNc].

Proposition 8.2. *For λ whose α_s component is zero,*

$$ \mathcal{T}_s({}_s\Delta(\lambda)) \cong \Delta_s(s(\lambda)). $$

Proof. Since $\mathcal{T}_s$ is monoidal and additive, by Proposition 7.4 it suffices to prove this for root modules. We will proceed by induction on the height of the root. First consider a root module for a minimal root - these are roots $\alpha \in \Phi^+ \setminus \{\alpha_s\}$ that cannot be written in the form $\beta + \gamma$ where β and γ are also in $\Phi^+ \setminus \{\alpha_s\}$. The proposition in this case follows from (4.1) and (4.2).

Now suppose that Δ is a root module for ${}_s\!\prec$ that is not minimal. Consider the short exact sequence from Lemma 7.3.

$$ 0 \to q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma \xrightarrow{f_{\beta\gamma}} \Delta_\gamma \circ \Delta_\beta \to \Delta^{\oplus m} \to 0. $$

So $\Delta^{\oplus m}$ is the cone of a nonzero morphism $f_{\beta\gamma}$ from $q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma$ to $\Delta_\gamma \circ \Delta_\beta$. Recall that $\mathrm{Hom}_R(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta) \cong k$. By Theorem 6.1, $\iota_\lambda(f_{\beta\gamma})$ spans the space

$$\mathrm{Hom}_{K(\mathcal{U})}(i_\lambda(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma), i_\lambda(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma)).$$

By induction on the height of Δ, $\mathcal{T}_s(\Delta)^{\oplus m}$ is the cone of $\mathcal{T}_s(f_{\beta\gamma})$ which is the unique up to scalar nonzero morphism in $\mathrm{Hom}_R(q^{-\beta\cdot\gamma}\mathcal{T}_i(\Delta_\beta \circ \Delta_\gamma), \mathcal{T}_i(\Delta_\gamma \circ \Delta_\beta)))$. Hence $\mathcal{T}_i(\Delta)^{\oplus m}$ is as desired and $\mathcal{T}_i(\Delta)$ is standard. □

Lemma 8.3. *Identify ${}_s\mathcal{C}$ and $\mathcal{C}_s$ with their essential images under i_λ and $i_{s\lambda}$. Let M be a module in ${}_s\mathcal{C}$ with a Δ-flag. Then $\mathcal{T}_s(M)$ lies in $\mathcal{C}_s$ and has a Δ-flag.*

Proof. We proceed by induction on the length of the Δ-flag of M. When this length is 1, this is Proposition 8.2. So now suppose that M has a Δ-flag of length greater than one and this result is known for all modules with a smaller Δ-flag than that of M.

Then there is a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where M' and M'' have a Δ-flag of smaller length than that of M.

The module M is identified with the cone of a morphism from $M''[-1]$ to M'. Since $\mathcal{T}_s$ is exact, $\mathcal{T}_s(M)$ is the cone of a morphism from $\mathcal{T}_s(M'')[-1]$ to $\mathcal{T}_s(M)$. By induction on the length of a Δ-flag, we know that $\mathcal{T}_s(M'')$ and $\mathcal{T}_s(M)$ are in $\mathcal{C}_s$. Therefore they are identified with a complex of projective R-modules, hence the same is true of $\mathcal{T}_s(M)$ since it appears as a cone. Now take the long exact seqence in homology associated to the triangle

$$\mathcal{T}_s(M'') \to \mathcal{T}_s(M) \to \mathcal{T}_s(M') \xrightarrow{+1} .$$

Since the homologies of $\mathcal{T}_s(M'')$ and $\mathcal{T}_s(M')$ are known to be concentrated in degree zero by our inductive hypothesis, the same is true of $\mathcal{T}_s(M)$. Hence $\mathcal{T}_s(M)$ is the class of a module, and appears as an extension of two modules with Δ-flags. Therefore it lies in $\mathcal{C}_s$ and has a Δ-flag. □

Consider a projective P in ${}_s\mathcal{C}$. Since we have shown that P has a Δ-flag and that standard modules get sent to standard modules, we know that $\mathcal{T}_s(P)$ lies in $\mathcal{C}_s$.

Lemma 8.4. *Let P be projective in ${}_s\mathcal{C}$ and let Δ be a standard module in $\mathcal{C}_s$. Then $\mathrm{Ext}^i_R(\mathcal{T}_s(P), \Delta) = 0$ for all $i > 0$.*

Proof. There are isomorphisms

$$\mathrm{Ext}^i_R(\mathcal{T}_s(P), \Delta) \otimes \mathcal{B}_\lambda \cong \mathrm{Ext}^i_R(P, \mathcal{T}_s^{-1}(\Delta)) \otimes \mathcal{B}_{s\lambda}.$$

As $\mathcal{T}_s^{-1}$ sends standard modules to standard modules, this is zero by Lemma 7.8. □

Lemma 8.5. *Let P be a projective object in ${}_s\mathcal{C}$. Then $\mathcal{T}_s(P)$ is projective in $\mathcal{C}_s$.*

Proof. Let L be a simple module in $\mathcal{C}_s$ and consider the short exact sequence

$$0 \to K \to \Delta \to L \to 0.$$

Apply $\mathrm{Hom}(T_s(P), -)$ and consider the corresponding long exact sequence of Ext groups. By Lemma 8.4, for $i > 0$, we obtain an isomorphism

$$\mathrm{Ext}^i(\mathcal{T}(P), L) \cong \mathrm{Ext}^{i+1}(\mathcal{T}(P), K).$$

As $\mathcal{T}_s(P)$ has a Δ-flag and R is Noetherian, $\mathcal{T}_s(P)$ is finitely presented. Thus for any simple module M, there are only a finite number of integers d for which $\mathrm{Ext}^i(\Delta, q^d M) \neq 0$.

All simple subquotients of K are less than L or a positive grading shift of L. We can induct on L and its grading shift to show that these Ext groups vanish, using the observation in the previous paragraph as the base case for the induction. □

Proof of Theorem 8.1. Identify ${}_s\mathcal{C}$ and $\mathcal{C}_s$ with their essential images under the faithful functors i_λ and $i_{s(\lambda)}$. Let P be projective in ${}_s\mathcal{C}$. Then by Lemma 8.5, there is a projective Q in $\mathcal{C}_s$ such that $\mathcal{T}_s(i_\lambda(P)) \cong i_{s\lambda}(Q)$. Since $\mathcal{T}_s$ is an equivalence, by Theorem 6.1, there is an induced isomorphism

$$\mathrm{End}_R(P) \otimes \mathcal{B}_\lambda \cong \mathrm{End}_R(Q) \otimes \mathcal{B}_{s_\lambda}.$$

As $\mathcal{T}_s$ also induces an isomorphism $\mathcal{B}_\lambda \cong \mathcal{B}_{s_\lambda}$, we get an induced isomorphism

$$End_R(P) \cong End_R(Q).$$

An abelian category is governed by the endomorphism algebra of a projective generator. Since $\mathcal{T}_s^{-1}$ induces an analogous isomorphism, it must be that $\mathcal{T}_s$ induces an equivalence of categories ${}_s\mathcal{C} \cong \mathcal{C}_s$, as required. □

9. RESTRICTION OF CATEGORICAL REPRESENTATIONS

We remind the reader that Φ is simply laced, and of finte or affine type.

Definition 9.1. *A face is a decomposition of Φ^+ into three disjoint subsets*

$$\Phi^+ = F^+ \sqcup F \sqcup F^-$$

such that, for all $x \in \mathrm{span}_{\mathbb{R}_{\geq 0}} F$:

(1) *If $y \in \mathrm{span}_{\mathbb{R}_{\geq 0}} F^+$ is non-zero, then $x + y \notin \mathrm{span}_{\mathbb{R}_{\geq 0}}(F^- \cup F)$.*
(2) *If $y \in \mathrm{span}_{\mathbb{R}_{\geq 0}} F^-$ is non-zero, then $x + y \notin \mathrm{span}_{\mathbb{R}_{\geq 0}}(F^+ \cup F)$.*

Example 9.2. *Suppose Φ^+ is of type $A_n^{(1)}$ and let $1 \leq e \leq n$ be an integer. Let $\lambda : \mathbb{R}\Phi^+ \longrightarrow \mathbb{R}$ be a generic linear map subject to the conditions*

$$\lambda(\alpha_0) = \lambda(\alpha_1) = \cdots = \lambda(\alpha_{e-1}) = 0 = \lambda(\alpha_e + \alpha_{e+1} + \cdots + \alpha_n)$$

and $\lambda(\alpha_e) < \cdots < \lambda(\alpha_n)$.

Then $F = \lambda^{-1}(0) \cap \Phi^+$ is a face of type $A_e^{(1)}$.

We choose this example because it appears in [Mak] and [RW]. Also it is not conjugate to a standard face under the Weyl group so the 2-functor of Theorem 9.4 cannot be obtained as a composition of reflection functors.

We will need the following alternative viewpoint on faces:

Theorem 9.3. [TW, Lemma 1.10] *For any face $\Phi^+ = F^+ \sqcup F \sqcup F^-$, there is a sequence of linear functionals $\{\lambda_n\}_{n \in \mathbb{N}}$ on $\mathbb{R}\Phi$ such that*

(1) $F \subset \ker(\lambda_n)$ *for all* n,
(2) *For all* $\alpha \in F^+$, $\lambda_n(\alpha) > 0$ *for all* $n \gg 0$,
(3) *For all* $\alpha \in F^-$, $\lambda_n(\alpha) < 0$ *for all* $n \gg 0$.

If the root system Φ is of finite type, or if the root system is of symmetric affine type and the field k is of characteristic zero, then to each face F, a KLR algebra R_F is constructed in [MT]. This is a KLR algebra for the face root system Φ_F. There is a choice of polynomials Q_{ij} to define this face KLR algebra which is implicitly determined in [MT]. With the same choice of polynomials, we can define the face 2-category $\mathcal{U}_F$.

WARNING: One may be worried that these choices are not related by the reflection functors $\mathcal{T}_i$. This is a valid concern which possibly means that some results below only hold up to scalar. I need to check this.

This construction is actually more general than as stated in the paragraph above, see [MT, Assumption 3.11] for a precise statement. The restrictions on Φ and the characteristic of k ensure that a theory of standard modules exists. However, this construction does not require the full theory of standard modules, but only for the theory of the root modules $\Delta(\alpha)$ for each $\alpha \in \Delta_F$. Such a theory of standard modules exists for the face of Example 9.2 in all characteristics, as mentioned in [MT, Assumption 3.11]. Therefore we can meaningfully talk about $\mathcal{U}_F$ and its offshoots in this example.

Let (F^-, F, F^+) be a face. Supopse $s \in S$ is such that the corresponding simple root α_s satisfies $\alpha_s \in F^-$. Then we can define a new face

$$\sigma_s(F^-, F, F^+) = (s(F^-) \setminus \{-\alpha_s\}, s(F), s(F^+) \sqcup \{\alpha_s\}).$$

Conversely if $\alpha_s \in F^+$ then there is a new face

$$\sigma_s^*(F^-, F, F^+) = (s(F^-) \sqcup \{\alpha_s\}, s(F), s(F^+) \setminus \{-\alpha_s\}).$$

In terms of the sequence of linear functionals $\{\lambda_n\}_{n\in\mathbb{N}}$ from Theorem 9.3, both of these constructions arise from the sequence $\{s(\lambda_n)\}_{n\in\mathbb{N}}$. If $\alpha_s \in F$, then applying s to each of the functionals in this seqeuence does not change the face.

Theorem 9.4. *There is a 2-functor* $K^b(\mathcal{U}_F) \to K^b(\mathcal{U})$.

Proof. The one-colour relations all follow from [ALELR] once we have our observation that the $\mathcal{T}_i$ send root modules to root modules. The quiver Hecke relations are all checked in [MT]. It remains to show that a rightward crossing with two different colours is invertible.

If the face is of affine type, then for a non-trivial face, the initial root system must be simply laced, hence every root is in the W-orbit of α_0. Let $\alpha \in \Delta_F$. Pick $w \in W$ such that $w\alpha = \alpha_0$. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of linear functionals associated to F from Theorem 9.3. We apply w to the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ to get a new face related by a sequence of reflection functors $\mathcal{T}_i$ and $\mathcal{T}_i^*$. Therefore we may assume without loss of generality that any given root $\alpha \in \Delta_F$ is α_0. So given two roots $\alpha, \beta \in \Delta_0$, without loss of generality $\alpha = \alpha_0$. Then since $\alpha + \beta$ is a positive summand of δ and the coefficient of α_0 in δ is 1, there is no occurrence of α_0 in β. Thus a projective resolution of $\Delta(\beta)$ by standard projective modules has no occurrence of E_0. So the map $E_0\Delta(\beta) \to \Delta(\beta)E_0$ as a map of complexes consists of all rightward crossings. It is then clear that the inverse of this map of complexes is the corresponding map with all leftward crossings.

If the face is of finite type then either F^- or F^+ is finite. WLOG F^- is finite. Apply a finite number of Saito reflections to reduce to the case when F^- is the empty set. Now there is a functional λ such that $\lambda(\alpha) \geq 0$ for all $\alpha \in \Phi^+$, and $\ker(\lambda) \cap \Phi^+ = F$. Then it is clear that F is a standard face, i.e. arising from an inclusion $J \subset I$, and in this case the result is obvious. □

It is natural to make the following conjecture:

Conjecture 9.5. *This 2-functor from $K^b(\mathcal{U}_F)$ to $K^b(\mathcal{U})$ is faithful.*

Remark 9.6. In finite type the face 2-functors $K^b(\dot{\mathcal{U}}_F) \to K^b(\dot{\mathcal{U}})$ are all compositions of the equivalences $\mathcal{T}_s$ and $\mathcal{T}_s^{-1}$, together with the inclusions of a standard face, and so in particular are faithful (the inclusions of standard faces are faithful by Webster's nondegeneracy theorem).

As a consequence, any time $\mathcal{U}$ acts on a category $\mathcal{A}$, we can restrict along the face to obtain an action of $\mathcal{U}_F$ on $K^b(\mathcal{A})$. In special cases (such as those considered in [Mak] and [RW]), we actually get an action of $\mathcal{U}_F$ on $\mathcal{A}$. It would be interesting to have an elegant and practical criterion to determine when this restricted action is an action on $\mathcal{A}$ instead of only an action on $K^b(\mathcal{A})$. In lieu of a beautiful criterion, we now give a necessary condition and show how we can use it to reconstruct the categorical restrictions of [Mak, RW].

For all $\alpha \in \Delta_F$, choose a projective resolution of $\Delta(\alpha)$. For each indecomposable projective appearing in a resolution except for those in homological degree zero, choose an inclusion $P \subset P_{\mathbf{i}}$ as a direct summand. Write $\mathbf{i} = (i_1, \ldots, i_n)$. Then we need, that for all weights λ and for all such $\mathbf{i}$, that at least one of the categories

$$\mathcal{A}(\lambda), \mathcal{A}(\lambda + i_1), \ldots, \mathcal{A}(\lambda + i_1 + \cdots + i_n)$$

is zero.

This works because it implies that for each generating 1-morphism of $K(\mathcal{U}_F)$, there is an endofunctor of $\mathcal{A}$ inducing the same action on $K(\mathcal{A})$.

This condition can be checked at the level of Grothendieck groups. For example, there is a categorical action of $\hat{\mathfrak{sl}}_p$ on the principal block of $\operatorname{Rep}(GL_n; \overline{\mathbb{F}}_p)$ for $n \geq p$, constructed in [RW]. At the level of Grothendieck groups, this categorifies the n-th exterior power of the natural representation of $\hat{\mathfrak{sl}}_p$ on $\mathbb{C}^p \otimes \mathbb{C}[t, t^{-1}]$.

References

[ALELR] Abram, Lamberto-Egan, Lauda, and Rose.

[BKM] Jonanthan Brundan, Alexander Kleshchev, and Peter J. McNamara. Homological properties of finite type Khovanov-Lauda-Rouquier algebras. arXiv:1210.6900.

[Bru] J. Brundan. On the definition of Kac-Moody 2-category. arXiv:1501.00350.

[CR08] Joseph Chuang and Raphaël Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. *Ann. of Math. (2)*, 167(1):245–298, 2008.

[K1] Syu Kato. PBW bases and KLR algebras.

[K2] S. Kato. On the monoidality of the Saito reflection functors.

[KL10] Mikhail Khovanov and Aaron D. Lauda. A categorification of quantum sl(n)[1]. *Quantum Topol.*, 1(1):1–92, 2010.

[Mak] Ruslan Maksimau. Categorical representations, KLR algebras and Koszul duality. arXiv:1512.04878.

[McN15] Peter J. McNamara. Finite dimensional representations of Khovanov-Lauda-Rouquier algebras I: Finite type. *J. Reine Angew. Math.*, 707:103–124, 2015.

[McNa] P. J. McNamara. Representation theory of geometric extension algebras. arXiv:1701.07949.

[McNb] Peter J. McNamara. Representations of Khovanov-Lauda-Rouquier algebras III: Symmetric affine type. arXiv:1407.7304.

[McNc] P. J. McNamara. Monidality of Kato's reflection functor.

[MT] P. J. McNamara and P. Tingley. Face functors for KLR algebras. arXiv:1512.04458.

[Rou12] Raphaël Rouquier. Quiver Hecke algebras and 2-Lie algebras. *Algebra Colloq.*, 19(2):359–410, 2012.

[RW] Simon Riche and Geordie Williamson. Tilting modules and the p-canonical basis. arXiv:1512.08296.

[TW] P. Tingley and B. Webster. Mirkovic-Vilonen polytopes and Khovanov-Lauda-Rouquier algebras arXiv:1210.6921

[Web] B. Webster. Unfurling Khovanov-Lauda-Rouquier algebras. arXiv:1603.06311.

[XZ] J. Xiao and M. Zhao. Geometric realizations of Lusztig's symmetries. arXiv:1501.01778.

[Zha] M. Zhao. Geometric realizations of Lusztig's symmetries of symmetrizable quantum groups. arXiv:1510.00937.

Email address: maths@petermc.net

[1]This appeared on the arXiv under the title *A diagrammatic approach to categorification of quantum groups III*