ON A BRAID GROUP ACTION

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Abstract. We discuss some consequences of the braid group action on a categorified quantum group. Results include a description of reflection functors for quiver Hecke algebras and a theory of restricting categorical representations along a face.

1. Introduction

It is a classical construction to associate to any symmetrisable Cartan matrix a quantum group $U_q(\mathfrak{g})$. We concern ourselves with a categorified version of this construction, where a strict 2-category $\mathcal{U}$ is produced whose Grothendieck group is canonically identified with the idempotented form of the corresponding quantum group.

This theory of a categorical quantum group originates with the work of Chuang and Rouquier [CR08] who used the notion of a categorical $\mathfrak{sl}_2$ action to construct interesting derived equivalences. The generalisation to general $\mathfrak{g}$ came in the work of Khovanov and Lauda [KL10] and Rouquier [Rou12]. They give two different presentations of a 2-category $\mathcal{U}$, the equivalence of which was shown by Brundan [Bru].

The braid group acts by algebra automorphisms on the quantum group $U_q(\mathfrak{g})$. The starting point of this paper is the lifting of this braid group action to a braid group action on the homotopy category $K(\mathcal{U})$ by autoequivalences $\mathcal{T}_s$. This action was recently constructed by [ALLR] under the assumption that the Cartan matrix is simply laced. In this paper we explore the implications of the exitence of this action. The applications which we study all require the theory of standard modules for KLR algebras. This theory was developed in [BKM] in finite type and in [McNb] in affine type over a field of characteristic zero. The necessary facts from this theory are all recalled when they are needed. We also prove that these standard modules are compatible with these autoequivalences $\mathcal{T}_s$ in a precise way in Proposition 7.2.

The first application is to the construction of reflection functors for KLR algebras. These are functors which categorify the Satio reflection on the crystal $B(\infty)$, as well as Lusztig's braiding automorphism T_s, restricted to the positive part of the quantum group. These T_s induce isomorphisms $\ker({}_sr) \cong \ker(r_s)$ inside $U_q(\mathfrak{g})^+$. Both subspaces $\ker({}_sr)$ and $\ker(r_s)$ are categorified by a Serre subcategory of the category of quiver Hecke modules, which we denote by ${}_s\mathcal{C}$ and $\mathcal{C}_s$ respectively. We show how the autoequivalence $\mathcal{T}_s$ induces an equivalence of the abelian categories ${}_s\mathcal{C}$ and $\mathcal{C}_s$.

This equivalence was obtained geometrically in finite simply laced type over a field of characteristic zero in [K]. This was subsequently generalised to finite simply laced type in all

Date: October 23, 2017.

characteristics in [McNa]. The related work of [XZ, Zha] provides a geometric incarnation of T_s in all types but does not produce results of the same strength that we provide here.

The second application is to a theory of restricting a categorical representation. In [MT], the notion of a face of a root system is introduced. Each face defines a root system and hence there is a KLR algebra associated to that face, which we will call R_F. The main result of [MT] is the construction of a fully faithful functor from R_F-mod to R-mod, whose essential image is explicitly determined in terms of a subcategory of cuspidal modules if the face root system is of finite type. In this paper we go beyond the results of [MT] and show that each face induces a functor between the homotopy categories of categorified quantum groups. This allows us to restrict a categorical action on a category $\mathcal{C}$ to a categorical action for the face quantum group on the homotopy category $K(\mathcal{C})$. In certain cases the restricted categorical action is actually an action on $\mathcal{C}$ and we give a criterion for checking this.

2. The 2-category

Let $\mathcal{B}_\lambda$ be the commutative k-algebra freely generated by all bubbles.

[KL10, Definition 3.15] is a definition of nondegenerate.

Theorem 2.1 (Webster's nondegeneracy theorem)**.** [Web, Theorem A] *Basis of Hom spaces between products of $\mathcal{E}$s and $\mathcal{F}$s is as large as possible. (i.e. $\mathcal{U}$ is nondegenerate)*

Out of $\mathcal{U}$, for $* \in \{b, +, -, \emptyset\}$ we construct a new 2-category $K^*(\dot{\mathcal{U}})$. Like $\mathcal{U}$, its objects are elements of P. For any two elements $\lambda, \mu \in P$, the morphism category in $K^*(\dot{\mathcal{U}})$ is defined to be

$$\mathrm{Hom}_{K^*(\dot{\mathcal{U}})}(\lambda, \mu) = \mathrm{Kar}(K^*(\mathrm{Hom}_{\mathcal{U}}(\lambda, \mu)))$$

where Kar refers to taking the Karoubian envelope.

3. Summary of Necessary Results

In [ALLR], a triangulated autoequivalence of $K^b(\dot{\mathcal{U}})$ is constructed for each $s \in S$. We denote by $\mathcal{T}_s$ the equivalence denoted $\mathcal{T}'_{s,1}$ in that paper. It satisfies the following properties:

If s and t are connected by an edge, then

$$\mathcal{T}_s(\mathcal{F}_t 1_\lambda) = q\mathcal{F}_t\mathcal{F}_s 1_{s\lambda} \to \clubsuit\mathcal{F}_s\mathcal{F}_t 1_{s\lambda}, \tag{3.1}$$

$$\mathcal{T}_s(q\mathcal{F}_s\mathcal{F}_t 1_\lambda \to \clubsuit\mathcal{F}_t\mathcal{F}_s 1_\lambda) = \mathcal{F}_t 1_{s\lambda} \tag{3.2}$$

where the differential in each case is the downward crossing. The symbol $\clubsuit$ denotes homological degree zero.

Theorem 3.1. [ALLR] *$\mathcal{T}_s$ decategorifies to T_s.*

4. Quiver Hecke Algebras

The KLR algebra R is the algebra generated by the upward strands, subject to the isotopy and quiver Hecke relations.

For $\nu \in \mathbb{N}I$, let $R(\nu)$ be the subalgebra of R consisting of diagrams where the colours of the strands add to ν.

By placing strands next to each other there is a nonunital inclusion of algebras $R(\lambda)\otimes R(\mu)\hookrightarrow R(\lambda+\mu)$. There is thus a corresponding induction functor

$$\mathrm{Ind}: R(\lambda)\text{-mod}\times R(\mu)\text{-mod}\longrightarrow R(\lambda+\mu)\text{-mod}$$

written $(M,N)\mapsto M\circ N$, given by

$$M\circ N := R(\lambda+\mu)e\otimes_{R(\lambda)\otimes R(\mu)}(M\otimes N)$$

where e is the image of the unit under the algebra inclusion.

Let $P_{\mathbf{i}_1\cdots\mathbf{i}_n}=R(\mathbf{i})e(\mathbf{i})=P_{\mathbf{i}_1}\circ\cdots\circ P_{\mathbf{i}_n}$. This is a projective $R(\mathbf{i})$-module.

There is the adjunction [McN15]

$$\mathrm{Ext}^i(A,B\circ C)\cong\mathrm{Ext}^i(\mathrm{Res}\,A,C\boxtimes B). \tag{4.1}$$

5. The embeddings of categories

For each $\lambda\in P$, there is a functor

$$K^-(R(\nu)\text{-pmod})\xrightarrow{i_\lambda}\mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda,\lambda+\nu).$$

This functor sends the projective $R(\nu)$-module $P_{\mathbf{i}_1\cdots\mathbf{i}_n}$ to $\mathcal{F}_{\mathbf{i}_1}\cdots\mathcal{F}_{\mathbf{i}_n}1_\lambda$

These functors satisfy a compatibility between induction and composition.

$$\begin{tikzcd}
K^-(R(\mu)\text{-pmod})\times K^-(R(\nu)\text{-pmod}) \arrow[r, "\mathrm{Ind}"] \arrow[d, "i_{\lambda-\nu}\times i_\lambda"] & K^-(R(\mu+\nu)\text{-pmod}) \arrow[d, "i_\lambda"] \\
\mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda-\nu,\lambda-\mu-\nu)\times\mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda,\lambda-\nu) \arrow[r] & \mathrm{Hom}_{K^-(\dot{\mathcal{U}})}(\lambda,\lambda-\mu-\nu)
\end{tikzcd}$$

where the horizontal map along the last row is the composition in $K^-(\dot{\mathcal{U}})$.

For each $\lambda\in P$, let $\mathcal{B}_\lambda$ be the endomorphism ring of the unit in the monoidal category $\mathrm{Hom}_{\mathcal{U}}(\lambda,\lambda)$. This space consists purely of bubbles. In degree zero, it is spanned by the identity and in negative degrees, it is zero.

Theorem 5.1. *Suppose X and Y are two objects in $K^-(R(\nu))$-pmod). Then there is an isomorphism of graded vector spaces.*

$$\mathrm{Hom}(i_\lambda(X),i_\lambda(Y))\cong\mathrm{Hom}(X,Y)\otimes\mathcal{B}_\lambda.$$

Proof. Since X and Y are bounded above, we can choose bounded above chain complexes $P_\bullet$ and $Q_\bullet$ for X and Y respectively such that each module P_i and Q_i is a direct sum of standard projective modules $R(\nu)e_{\mathbf{i}}$.

Let $\{h_j\}$ be a basis of the space $\mathcal{B}_\lambda$.

Let $f_\bullet: i_\lambda(P_\bullet)\longrightarrow i_\lambda(Q_\bullet)$ be a morphism of chain complexes. We can express each f_i in the form $f_i=\sum_j g_{ij}\otimes h_j$ where each g_{ij} has no bubbles, i.e. comes from a morphism in $K^-(R(\nu)$-mod).

That $f_\bullet$ is a chain map is expressed in the identity

$$\sum_j dg_{ij}\otimes h_j=\sum_j g_{i+1,j}d\otimes h_j.$$

By Webster's nondegeneracy theorem, this implies that for each j, $dg_{ij} = g_{i+1,j}d$. Thus, the collection $g_{\bullet,j}$ is a chain map in $K^-(R(\nu)\text{-pmod})$.

This shows that the map from $\mathrm{Hom}(X,Y) \otimes \mathcal{B}_\lambda$ to $\mathrm{Hom}(i_\lambda(X), i_\lambda(Y))$ is surjective. A similar argument shows that if f is homotopic to zero, then this homotopy comes from homotopies between each g_{ij} and zero. Therefore this map is injective also. □

Corollary 5.2. *The functor i_λ is faithful.*

6. Standard modules

We summarise the current state of the theory of standard modules for quiver Hecke algebras. This theory is currently known to exist in finite type in all characteristics and in symmetric affine type when k is of characteristic zero. The references are [BKM] in the former case and [McNb] in the latter.

Let Φ^+ be the set of positive roots. A *convex order* on Φ^+ is a preorder $\prec$ such that

- If S and T are two subsets of Φ^+ such that $s \prec t$ for all $s \in S$ and $t \in T$ then
$$\mathrm{span}_{\mathbb{R}_{\geq 0}} S \cap \mathrm{span}_{\mathbb{R}_{\geq 0}} T = \{0\},$$
- If $s \preceq t$ and $t \preceq s$ then s and t are proportional.

Let $\alpha \in \Phi^+$. A representation M of $R(\alpha)$ is said to be semicuspidal (with respect to the convex order $\prec$) if $\mathrm{Res}_{\beta\gamma} M \neq 0$ implies that β is a sum of roots less than α and γ is a sum of roots greater than α.

Let α be an indivisible root. An indecomposable projective object in the category of semicuspidal $R(\alpha)$-modules is called a root module. The grading shift on these root modules is customarily normalised such that their heads are self-dual. For each indecomposable root α, the number of root modules for α is equal to the dimension of the root space $\mathfrak{g}_\alpha$. In particular, if α is a real root, there is a unique root module, which we call $\Delta(\alpha)$.

We consider the standard modules introduced in [BKM] and [McNb]. These depend on the convex order $\prec$ and are built out of root modules. The root modules corresponding to real roots have already been introduced, these are the modules $\Delta(\alpha)$. For the indivisible imaginary root δ, we will call the modules denoted $\Delta(\omega)$ in [McNb] root modules. These are the projective modules in the category of cuspidal $R(\delta)$-modules.

Standard modules are naturally indexed by root partitions. A root partition is a sequence $\lambda = (\alpha_1^{n_1}, \cdots, \alpha_l^{n_l})$ where $\alpha_1 \succ \cdots \succ \alpha_l$ are indivisible roots, each n_i is a positive integer unless $\alpha_i = \delta$, in which case it is a collection of partitions. To each term $\alpha_i^{n_i}$ a standard module $\Delta(\alpha_i)^{(n_i)}$ is constructed. If α_i is real then $\Delta(\alpha_i)^{\circ n_i}$ is a direct sum of $n_i!$ copies of the module $\Delta(\alpha_i)^{(n_i)}$ with grading shifts. If α_i is imaginary then $\Delta(\alpha_i)^{(n_i)}$ is a summand of a product of certain modules $\Delta(\omega)$ of weight δ in $\mathcal{C}_F$; see [McNb] for the details (where this module is denoted $\Delta(\underline{\lambda})$). The standard module is then defined to be the indecomposable module
$$\Delta(\lambda) = \Delta(\alpha_1)^{(n_1)} \circ \cdots \circ \Delta(\alpha_l)^{(n_l)}.$$
In [BKM] and [McNb] homological properties of these modules are developed which justify the use of the name standard.

If α is a real root, let $L(\alpha)$ be the head of $\Delta(\alpha)$ and let $L(\alpha^n) = L(\alpha)^{\circ n}$. If $\underline{\lambda}$ is a multipartition, let $L(\underline{\lambda})$ be the head of $D(\underline{\lambda})$. These modules are all simple.

Let $\lambda = (\alpha_1^{n_1}, \cdots, \alpha_l^{n_l})$ be a root partition. Define the proper costandard module

$$\overline{\nabla}(\lambda) = L(\alpha_l^{n_l}) \circ \cdots \circ L(\alpha_1^{n_1}).$$

Theorem 6.1. *The simple modules for $R(\nu)$ are classified up to isomorphism and grading shift by root partitions. The simple module $L(\lambda)$ corresponding to the root partition λ is the socle of $\overline{\nabla}(\lambda)$. Furthermore every other simple subquotient $L(\mu)$ of $\overline{\nabla}(\lambda)$ has $\lambda \prec \mu$ in the lexicographical order on root partitions.*

Proposition 6.2. [McNb, Proposition 24.3] *Let Δ be a standard module and $\overline{\nabla}$ be a proper standard module. Then for $i > 0$,*

$$\mathrm{Ext}^i(\Delta, \overline{\nabla}) = 0.$$

Lemma 6.3. *Let Δ be a root module. Then there are root modules Δ_β and Δ_γ and a nonzero q-integer m such that there is a short exact sequence*

$$0 \to q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma \xrightarrow{f_{\beta\gamma}} \Delta_\gamma \circ \Delta_\beta \to \Delta^{\oplus m} \to 0.$$

If $wt(\Delta)$ is not minimal in $\Phi^+ \setminus \{\alpha_s\}$, then Δ_β and Δ_γ can be chosen to be in $\Phi^+ \setminus \{\alpha_s\}$. Furthermore $f_{\beta\gamma}$ spans $\mathrm{Hom}(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta)$ and $\mathrm{HOM}(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta)$ is concentrated in nonnegative degrees.

Proof. In finite type, this is [BKM, Theorem 4.10]. In symmetric affine type, this is [McNb, Lemma 16.1] if $\mathrm{wt}(\Delta)$ is real and [McNb, Theorem 17.1] if $\mathrm{wt}(\Delta)$ is imaginary. The statement about $\mathrm{Hom}(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta)$ being one-dimensional is not explcitly mentioned in these references, but is clear from the proofs. Note that we can replace $\Phi^+ \setminus \{\alpha_s\}$ by any interval. □

Proposition 6.4. *Every standard module is obtained from a root module by a process of induction and taking direct summands.*

Remark 6.5. We expect that if a theory of standard modules is developed in affine type over a field of positive characteristic, it will not satisfy Proposition 6.4.

Definition 6.6. *A module M is said to have a Δ-flag if there exists a filtration by submodules*

$$M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

such that each subquotient M_{i+1}/M_i is a standard module.

Lemma 6.7. [BKM, Theorem 3.13] *A finitely generated module M has a Δ-flag if and only if $\mathrm{Ext}^1(M, \overline{\nabla}) = 0$ for all proper costandard modules $\overline{\nabla}$.*

Pick a convex order ${}_s\prec$ which has α_s as its largest element. From ${}_s\prec$ we can obtain another convex order $\prec_s$, by

- $\alpha_s \prec_s \beta$ for all $\beta \in \Phi^+ \setminus \{\alpha_s\}$
- $\beta \prec_s \gamma$ if $s\beta \,{}_s\!\prec s\gamma$ for all $\beta, \gamma \in \Phi^+ \setminus \{\alpha_s\}$

These convex orders induce two families of standard modules in R-mod, we denote them by ${}_s\Delta(\lambda)$ and $\Delta_s(\lambda)$.

Lemma 6.8. *Suppose P is projective in ${}_s\mathcal{C}$ (or in $\mathcal{C}_s$). Then $\mathrm{Ext}^i_R(P,M)=0$ for all R-modules M in ${}_s\mathcal{C}$ (respectively $\mathcal{C}_s$) and $i>0$.*

Note that the Ext group is computed in the category of R-modules rather than in ${}_s\mathcal{C}$ (or $\mathcal{C}_s$), so this result is not a tautology for $i>1$.

Proof. We consider the case where P is projective in ${}_s\mathcal{C}$, the other case following similarly. First we prove that P has a Δ-flag. By Lemma 6.7 it suffices to prove that $\mathrm{Ext}^1(P,\overline{\nabla})=0$ for all proper costandard modules $\overline{\nabla}$. If λ is a root partition with $\lambda(\alpha_s)=0$, then $\overline{\nabla}(\lambda)\in {}_s\mathcal{C}$. So in this case $\mathrm{Ext}^1(P,\overline{\nabla}(\lambda))=0$ as P is projective in ${}_s\mathcal{C}$. If $\lambda(\alpha_s)\neq 0$, then $\overline{\nabla}(\lambda)\cong X\circ L_s$ for some X. Then by (4.1),

$$\mathrm{Ext}^1(P,\overline{\nabla}(\lambda))\cong \mathrm{Ext}^1(\mathrm{Res}_{\alpha_s,\nu-\alpha_s}P, L_s\otimes X).$$

Since $P\in_s\mathcal{C}$, this restriction is zero and hence this Ext group is zero.

Therefore we have succeeded in showing that $\mathrm{Ext}^1(P,\overline{\nabla})=0$ for all proper costandard modules $\overline{\nabla}$, hence P has a Δ-flag. By Propsition 6.2, this implies that $\mathrm{Ext}^i(P,\overline{\nabla})=0$ for all proper costandard modules $\overline{\nabla}$ and $i>0$.

We now turn our attention to the statement that $\mathrm{Ext}^i_R(P,M)=0$ for all R-modules M in ${}_s\mathcal{C}$.

Without loss of generality we may assume M is simple. Then M injects into a proper costandard module $\overline{\nabla}$. Let Q be the quotient $\overline{\nabla}/M$. Then every simple subquotient L of Q satisfies $L\prec M$.

By induction on the partial preorder $\prec$, we may assume $\mathrm{Ext}^i(P,L)=0$ for all $L\prec M$ and $i>0$. Hence $\mathrm{Ext}^i(P,Q)=0$ for $i>0$. Now apply $\mathrm{Hom}(P,-)$ to the short exact sequence

$$0\to M\to\overline{\nabla}\to Q\to 0.$$

The resulting long exact sequence implies that $\mathrm{Ext}^i(P,M)=0$ for $i\geq 2$.

This leaves only the $i=0$ case, but $\mathrm{Ext}^1(P,M)$ vanishes since P is projective in ${}_s\mathcal{C}$ and M is in ${}_s\mathcal{C}$. ∎

7. Reflection Functor

Let R be a quiver Hecke algebra and $s\in S$. Let ${}_se$ be the sum of all generating idempotents with first strand coloured s. Let e_s be the sum of all generating idempotents with last strand coloured s. Let ${}_s\mathcal{C}$ (respectively $\mathcal{C}_s$) be the full subcategory of R-modules on which ${}_se$ (respectively e_s) acts by zero.

Equivalently

$${}_s\mathcal{C}=R/\langle {}_se\rangle\text{-mod}\quad\text{and}\quad \mathcal{C}_s=R/\langle e_s\rangle\text{-mod}.$$

Fix $s\in S$. For each s, we have inclusions of full subcategories

$${}_s\mathcal{C},\mathcal{C}_s\subset R\text{-mod}\subset K^-(R\text{-pmod}).$$

If our Cartan datum is of finite type then R has finite global dimension by [McN15, Theorem 4.7]. So in this case, the essential image lies in the bounded homotopy category $K^b(R\text{-pmod})$. In general, the standard modules have finite projective dimension so lie in $K^b(R\text{-pmod})$, while R in general has infinite global dimension.

Theorem 7.1. *Suppose our quiver Hecke algebra is simply laced, of finite or affine type. If we are in affine type, assume furthermore that the ground field is of characteristic zero. Then there is an equivalence of categories ${}_s\mathcal{C} \cong \mathcal{C}_s$ which decategorifies to Lusztig's braid group automorphism.*

The restriction to simply laced is due to the generality of [ALLR]. The further restrictions are needed since we rely on the theory of standard modules, developed in [BKM] (building on [McN15]) in finite type and in [McNb] in affine type over a field of characteristic zero.

In finite type ADE and in characteristic zero, Kato [K] has proved this equivalence geometrically. In finite type ADE and in all characteristics there is a geometric proof in [McNa] The paper of Xiao and Zhao [XZ] provides an interpretation of T_s in terms of perverse sheaves in all symmetric types which is subsequently generalised to all symmetrisable types in [Zha]. Their results are not as strong as the equivalence of categories in Theorem 7.1.

Proposition 7.2. *For λ whose α_s component is zero,*

$$\mathcal{T}_s({}_s\Delta(\lambda)) \cong \Delta_s(s(\lambda)).$$

Proof. Since $\mathcal{T}_s$ is monoidal and additive, by Proposition 6.4 it suffices to prove this for root modules. We will proceed by induction on the height of the root. First consider a root module for a minimal root - these are roots $\alpha \in \Phi^+ \setminus \{\alpha_s\}$ that cannot be written in the form $\beta + \gamma$ where β and γ are also in $\Phi^+ \setminus \{\alpha_s\}$. The proposition in this case follows from (3.1) and (3.2).

Now suppose that Δ is a root module for ${}_s\prec$ that is not minimal. Consider the short exact sequence from Lemma 6.3.

$$0 \to q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma \xrightarrow{f_{\beta\gamma}} \Delta_\gamma \circ \Delta_\beta \to \Delta^{\oplus m} \to 0.$$

So $\Delta^{\oplus m}$ is the cone of a nonzero morphism $f_{\beta\gamma}$ from $q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma$ to $\Delta_\gamma \circ \Delta_\beta$. Recall that $\mathrm{Hom}_R(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma, \Delta_\gamma \circ \Delta_\beta) \cong k$. By Theorem 5.1, $\iota_\lambda(f_{\beta\gamma})$ spans the space

$$\mathrm{Hom}_{K(\mathcal{U})}(i_\lambda(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma), i_\lambda(q^{-\beta\cdot\gamma}\Delta_\beta \circ \Delta_\gamma)).$$

By induction on the height of Δ, $\mathcal{T}_s(\Delta)^{\oplus m}$ is the cone of $\mathcal{T}_s(f_{\beta\gamma})$ which is the unique up to scalar nonzero morphism in $\mathrm{Hom}_R(q^{-\beta\cdot\gamma}\mathcal{T}_i(\Delta_\beta \circ \Delta_\gamma), \mathcal{T}_i(\Delta_\gamma \circ \Delta_\beta)))$. Hence $\mathcal{T}_i(\Delta)^{\oplus m}$ is as desired and $\mathcal{T}_i(\Delta)$ is standard. □

Lemma 7.3. *Identify ${}_s\mathcal{C}$ and $\mathcal{C}_s$ with their essential images under i_λ and $i_{s\lambda}$. Let M be a module in ${}_s\mathcal{C}$ with a Δ-flag. Then $\mathcal{T}_s(M)$ lies in $\mathcal{C}_s$ and has a Δ-flag.*

Proof. We proceed by induction on the length of the Δ-flag of M. When this length is 1, this is Proposition 7.2. So now suppose that M has a Δ-flag of length greater than one and this result is known for all modules with a smaller Δ-flag than that of M.

Then there is a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

where M' and M'' have a Δ-flag of smaller length than that of M.

The module M is identified with the cone of a morphism from $M''[-1]$ to M'. Since $\mathcal{T}_s$ is exact, $\mathcal{T}_s(M)$ is the cone of a morphism from $\mathcal{T}_s(M'')[-1]$ to $\mathcal{T}_s(M)$. By induction on the length of a Δ-flag, we know that $\mathcal{T}_s(M'')$ and $\mathcal{T}_s(M)$ are in $\mathcal{C}_s$. Therefore they are identified with a complex of projective R-modules, hence the same is true of $\mathcal{T}_s(M)$ since it appears as a cone. Now take the long exact seqence in homology associated to the triangle

$$\mathcal{T}_s(M'') \to \mathcal{T}_s(M) \to \mathcal{T}_s(M') \xrightarrow{+1}.$$

Since the homologies of $\mathcal{T}_s(M'')$ and $\mathcal{T}_s(M')$ are known to be concentrated in degree zero by our inductive hypothesis, the same is true of $\mathcal{T}_s(M)$. Hence $\mathcal{T}_s(M)$ is the class of a module, and appears as an extension of two modules with Δ-flags. Therefore it lies in $\mathcal{C}_s$ and has a Δ-flag. □

Consider a projective P in ${}_s\mathcal{C}$. Since we have shown that P has a Δ-flag and that standard modules get sent to standard modules, we know that $\mathcal{T}_s(P)$ lies in $\mathcal{C}_s$.

Lemma 7.4. *Let P be projective in ${}_s\mathcal{C}$ and let Δ be a standard module in $\mathcal{C}_s$. Then $\mathrm{Ext}^i_R(\mathcal{T}_s(P), \Delta) = 0$ for all $i > 0$.*

Proof. There are isomorphisms

$$\mathrm{Ext}^i_R(\mathcal{T}_s(P), \Delta) \otimes \mathcal{B}_\lambda \cong \mathrm{Ext}^i_R(P, \mathcal{T}_s^{-1}(\Delta)) \otimes \mathcal{B}_{s\lambda}.$$

As $\mathcal{T}_s^{-1}$ sends standard modules to standard modules, this is zero by Lemma 6.8. □

Lemma 7.5. *Let P be a projective object in ${}_s\mathcal{C}$. Then $\mathcal{T}_s(P)$ is projective in $\mathcal{C}_s$.*

Proof. Let L be a simple module in $\mathcal{C}_s$ and consider the short exact sequence

$$0 \to K \to \Delta \to L \to 0.$$

Apply $\mathrm{Hom}(T_s(P), -)$ and consider the corresponding long exact sequence of Ext groups. By Lemma 7.4, for $i > 0$, we obtain an isomorphism

$$\mathrm{Ext}^i(\mathcal{T}(P), L) \cong \mathrm{Ext}^{i+1}(\mathcal{T}(P), K).$$

As $\mathcal{T}_s(P)$ has a Δ-flag and R is Noetherian, $\mathcal{T}_s(P)$ is finitely presented. Thus for any simple module M, there are only a finite number of integers d for which $\mathrm{Ext}^i(\Delta, q^d M) \neq 0$.

All simple subquotients of K are less than L or a positive grading shift of L. We can induct on L and its grading shift to show that these Ext groups vanish, using the observation in the previous paragraph as the base case for the induction. □

Proof of Theorem 7.1. Identify ${}_s\mathcal{C}$ and $\mathcal{C}_s$ with their essential images under the faithful functors i_λ and $i_{s(\lambda)}$. Let P be projective in ${}_s\mathcal{C}$. Then by Lemma 7.5, there is a projective Q in $\mathcal{C}_s$ such that $\mathcal{T}_s(i_\lambda(P)) \cong i_{s\lambda}(Q)$. Since $\mathcal{T}_s$ is an equivalence, by Theorem 5.1, there is an induced isomorphism

$$\mathrm{End}_R(P) \otimes \mathcal{B}_\lambda \cong \mathrm{End}_R(Q) \otimes \mathcal{B}_{s_\lambda}.$$

As $\mathcal{T}_s$ also induces an isomorphism $\mathcal{B}_\lambda \cong \mathcal{B}_{s_\lambda}$, we get an induced isomorphism

$$End_R(P) \cong End_R(Q).$$

An abelian category is governed by the endomorphism algebra of a projective generator. Since $\mathcal{T}_s^{-1}$ induces an analogous isomorphism, it must be that $\mathcal{T}_s$ induces an equivalence of categories ${}_s\mathcal{C} \cong \mathcal{C}_s$, as required. □

8. Restriction of categorical representations

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[1]This appeared on the arXiv under the title *A diagrammatic approach to categorification of quantum groups III*