SINGULARITIES OF SCHUBERT VARIETIES WITHIN A RIGHT CELL

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Abstract. We describe an algorithm which, given two permutations, produces two new permutations lying in the same right Kazhdan-Lusztig cell (of a bigger rank symmetric group). There is an isomorphism between the Richardson varieties corresponding to the two pairs of permutations which preserves the singularity type. This fact has applications in the study of W-graphs for symmetric groups, as well as in finding examples of reducible associated varieties of $\mathfrak{sl}_n$-highest weight modules, and comparing various bases of irreducible representations of the symmetric group or its Hecke algebra.

1. Introduction

Denote by $\mathcal{F}l_n$ the variety of complete flags in $\mathbb{C}^n$. The maximal torus T of diagonal matrices of GL_n acts on $\mathcal{F}l_n$ with fixed point set indexed by the permutations in S_n. The attractive sets of this action provide a cell decomposition of $\mathcal{F}l_n$. For a permutation $y \in S_n$, we denote by y, C_y, and X_y, the corresponding fixed point, Schubert cell and Schubert variety respectively. The Bruhat order on S_n is defined as $x \leq y$ if $C_x \subseteq X_y$.

Right cells were introduced by Kazhdan and Lusztig in their seminal paper [KL1] in order to construct representations of the Hecke algebra associated to a Coxeter group W. In general, right cells are equivalence classes on W defined using the Kazhdan-Lusztig basis of the Hecke algebra. When W is a symmetric group, there is a simple combinatorial description in terms of the Robinson-Schensted correspondence, namely that two permutations belong to the same right cell if and only if their P symbols coincide.

The main result of this paper is the following:

Theorem 1.1. *Let $x, y \in S_n$ with $x \leq y$. Then, there exist $N \geq n$, and two permutations $v, w \in S_N$ with $v \leq w$ such that*

- *v and w belong to the same right cell;*
- *the singularity type of X_w at v is the same as the singularity type of X_y at x.*

The permutations v, w of the above theorem are obtained from x and y by an explicit algorithm, described in Section 2. As a consequence,

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every singularity type that can appear in a type A Schubert variety can appear within a single right cell. Similarly, an analogous statement holds for left cells.

We have a couple of motivations for considering singularities which appear inside a single right cell.

One is in understanding which integers can arise as edge weights in a W-graph for a symmetric group. These integers are given by the coefficient of the highest possible degree monomial in the Kazhdan-Lusztig polynomials $P_{x,y}(q)$ where x and y belong to the same left cell, or equivalently, x^{-1} and y^{-1} belong to the same right cell. An old conjecture (the 0-1 conjecture) stated that such integers were always either 0 or 1. If x, y, v, w are as in Theorem 1.1, then $P_{x,y}(q) = P_{v,w}(q) = P_{v^{-1},w^{-1}}(q)$ and the highest possible degree is the same. It is hence enough to find a pair of permutations x, y not necessarily in the same cell with $\mu(x, y) > 1$ to produce a pair v^{-1}, w^{-1} of elements belonging to the same left cell and satisfying $\mu(v^{-1}, w^{-1}) > 1$ (see Example 2.5). The first example of a pair with such a property was exhibited in [MW] and relied on computer computations.

Another interest in finding the permutations $v, w \in S_N$ is motivated by Williamson's negative answer to a question by Borho-Brylinski and Joseph. We will comment more on their question in Section 4.1. For the moment, we only want to mention that the answer provided in [W] is obtained by using Howlett and Nguyen's software [HN] for computing W-graphs in magma [BCP]. With our algorithm, a variant of Williamson's example, can be obtained by performing in an elementary fashion. (see Example 2.4).

A further reason to look at singularities within a cell is to compare bases of Specht modules: For a partition λ of n, the Specht module S^λ has many different basis - the Springer basis, the Goldie rank basis, the Kazhdan-Lusztig basis and the p-Kazhdan-Lusztig bases for each prime p (see §4 for more details). Our algorithm allows us to transfer known examples of singularities of Schubert varieties inot a single cell and thus exhibit many examples where these bases differ.

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2. Robinson-Schensted Correspondence

The Robinson-Schensted correspondence is a bijection between S_n and pairs of standard tableaux of the same shape with n boxes. If $w \in$

S_n, we write $(P(w), Q(w))$ for the corresponding pair of tableaux. The P symbol $P(w)$ is obtained by successively performing n column insertions into the empty tableau, with numbers $w(n), w(n-1), \ldots, w(1)$ in that order. The equivalence of this with the more familiar row insertion definition follows from [S, Lemma 7.23.15].

The following is the right cell version of [A, Theorem A] or [GM, Fact 8], and is originally from [KL1, §5].

Theorem 2.1. *Two permutations x and y are in the same right cell if and only if they have the same P symbol.*

For a standard tableau T and an entry s of T, we denote by $c_T(s)$ the column index of the entry.

Lemma 2.2. *Let T be a standard tableau, and let s be an entry of T. Let r be such that no entries of T lie in the interval (r, s) and let k be a positive integer. Let T' be the tableau obtained from T by column inserting the $k-1$ numbers $r_{k-1}, \ldots, r_1$, where $r < r_1 < \cdots < r_{k-1} < s$. Then*

(1) T' has r_i in its i-th column and s in its m-th column, where $m = \max\{c_T(s), k\}$,
(2) if s' is an entry of T such that $s' > s$, then either $c_{T'}(s') = c_T(s')$ or $c_{T'}(s') \leq m + n$, where m is as before and n is the number of entries in T in $[s, s')$.

Proof. By induction on t, we show that if we column insert $r_{k-1}, \ldots, r_{k-t}$, then

- r_{k-t+i} lies in the $(i+1)$-th column for $0 \leq i < t$,
- s lies in the m-th column, for $m = \max\{c_T(s), t+1\}$.

The base case $t = 0$ is tautologically true. Now suppose we have inserted $r_{k-1}, \ldots, r_{k-t+1}$, in accordance with the inductive hypothesis. After these insertions, s will be placed in column $\max\{c_T(s), t\}$. When we column insert r_{k-t}, any r_{k-t+i} bumps out $r_{k-t+i+1}$ from the i-th column. In particular, r_{k-1} is placed in the t-th column and s gets bumped out if and only if $\max\{c_T(s), t\} = t$, that is $c_T(s) \leq t$. This proves the first part of the lemma.

We now prove the second statement by induction on n. Let $s_1 < \ldots < s_{n-1}$ be the elements of (s, s') that lie in T. During the column insertions s' can be bumped out only by the entries between r and s'. By the previous part of this Lemma, the entries $r_1, \ldots, r_{k-1}, s$ lie in the first m columns of T', and hence can bump s' out from its box only if $c_T(s') \leq m$ and none of the entries $s_1, \ldots, s_{n-1}$ appear in the first m columns. If this is the case, $c_{T'}(s') \leq m + 1$. Clearly, the entries $s_1, \ldots, s_{n-1}$ can bump s' out of its box only if they move. Hence, assume that s' gets bumped out from its box the first time by an s_i, then it will always be bumped out by it, unless it lands on a column which contains s_j for some j. If this is the case, then s' stays on that column.

We conclude that $c_{T'}(s') \leq c_{T'}(s_i)+1$, and hence the statement follows by induction. $\square$

Thanks to the the first part of the previous lemma, we are now able to prove the following central result.

Theorem 2.3. *Given $x, y \in S_n$, there exist $u, v \in S_N$ for some $N \geq n$ with $P(u) = P(v)$, $u(i) = v(i)$ for $i \leq N - n$ and with the pattern of u and v in the last n positions being the permutations x and y.*

Proof. Let k be the largest integer such that $P(x)$ and $P(y)$ have all entries less than or equal to k in the same place (If $P(x) = P(y)$ we set $k = n$). We perform an induction on the value of $n - k$, the case $n - k = 0$ being trivial. Define t by

$$t := \max\{c_{P(x)}(k+1), c_{P(y)}(k+1)\} - 1.$$

Let $n' = n + t$ and define $x', y' \in S_{n'}$ by

$$x'(i) = \begin{cases} k+i & \text{if} \quad i \leq t, \\ x(i-t) & \text{if} \quad i > t \text{ and } x(i-t) \leq k, \\ x(i-t)+t & \text{if} \quad i > t \text{ and } x(i-t) > k, \end{cases} \tag{2.1}$$

with y' defined similarly from y.

By the description of P in terms of column insertion, $P(x')$ and $P(y')$ are obtained by inserting the numbers $k+1$, $k+2$, $\ldots$, $k+t$, into tableau obtained from $P(x)$ and $P(y)$ by adding t to all entries greater than k. By Lemma 2.2(1), $P(x')$ and $P(y')$ agree for all entries less than or equal to $k + t + 1$.

Let k' be defined from x' and y' analogously to the definition of k. Then $n' - k' \leq (n + t) - (k + t + 1) < n - k$. So by induction there exists u and v with $P(u) = P(v)$ and the pattern of u and v in the last n' positions being the permutations x' and y'. Since the patterns of x' and y' in the last n positions are that of x and y, this u and v satisfy the conditions of the theorem, completing the proof. $\square$

We now give some examples, where we use the one-line notation to represent permutations.

Example 2.4. *If $x = [21654387]$ and $y = [62845173]$, then the proof outputs the pair $u = [895621a743cb]$ and $v = [8956a2c471b3]$, where $a = 10$, $b = 11$, $c = 12$.*

We choose this example because the singularity of X_y at x is the Kashiwara-Saito singularity [KS] and to point out that the pair $u, v \in S_{12}$ that we obtain is different from the permutations chosen in [W].

Example 2.5. *If $x = [edcbajihgf]$ and $y = [jeghbicdfa]$ then the proof outputs the pair $u = [xyz12rstuvmnophiefcjgdba3wqlk]$ and $v = [xyz12rstuvmnophiefc3jlqbwdgka]$. Here we identify S_{29} with the permutations of the set $\{a, b, c, \ldots, z, 1, 2, 3\}$.*

We choose this example because it appears in [MW]. Here the authors give the first examples of $x, y \in S_n$ such that the coefficient $\mu(x, y)$ of the highest possible degree term of the corresponding Kazhdan-Lusztig polynomial is > 1, and the first example of w and z in the same left cell with $\mu(w, z) > 1$.

Their x and y are the same as ours here, and $\mu(x, y) = 4$. The pair u^{-1}, v^{-1} is another instance of two permutations lying in the same left cell and having $\mu(u^{-1}, v^{-1}) = \mu(x^{-1}, y^{-1}) = \mu(x, y) = 4 > 1$. Thanks to our main result, this pair can be obtained without any computer calculation. The example for w and z provided in [MW] lies in S_{16} and has $\mu(u, v) = 5$.

Proposition 2.6. *In Theorem 2.3 above, we can always take* $N \leq n(n+1)/2$.

Proof. The inductive proof of Theorem 1.1 gives an algorithm for constructing u and v from x and y. It produces a sequence of permutations $\{x_i\}$, $\{y_i\}$ where $x_0 = x$, $y_0 = y$ and $x_{i+1} = x'_i$ and $y_{i+1} = y'_i$ are defined as in (2.1) where k_i+1 is the minimal entry which appears in a different place in $P(x_i)$ and $P(y_i)$, and $t_i = \max\{c_{P(x_i)}(k_i+1), c_{P(y_i)}(k_i+1)\}-1$. Let $n_i = n + \sum_{j=0}^{i-1} t_j$ be the index of the symmetric group x_i and y_i lie in.

We prove now by induction on $i \geq 1$ that for any $l > k_{i-1} + t_{i-1}$

$$c_{P(x_i)}(l), c_{P(y_i)}(l) \leq l - n_i + n.$$

Let $i = 1$. By Lemma 2.2(2), either $c_{P(x_1)}(l) = c_{P(x)}(l-t_0)$ or $c_{P(x_1)}(l) \leq (t_0 + 1) + l - (k_0 + 1 + t_0)$. In the first case, the thesis follows from $c_{P(x)}(l - t_0) \leq l - t_0 = l - n_1 + n$, as $P(x)$ is standard. As for the second case, we just notice that $k_0 \geq t_0$, and hence also in this case we have $c_{P(x_1)}(l) \leq l - t_0$.

The induction step is proven analogously: by Lemma 2.2(2) we can distinguish the two cases $c_{P(x_i)}(l) = c_{P(x_{i-1})}(l - t_{i-1})$ and $c_{P(x_i)}(l - t_{i-1}) \leq (t_{i-1} + 1) + l - (k_{i-1} + 1 + t_{i-1}) = l - k_{i-1}$. In the first case, the thesis follows by induction, while for the second case it is enough to observe that $k_{i-1} \geq n_i - n$, which follows inductively from $k_i \geq k_{i-1} + t_{i-1}$.

Clearly, the same upper bound is obtained for $c_{P(y_i)}(m)$. Since $k_i + 1 > k_{i-1} + t_{i-1}$, we get

$$c_{P(x_i)}(k_i + 1), c_{P(y_i)}(k_i + 1) \leq k_i + 1 - n_i + n.$$

Therefore $t_i \leq n - (n_i - k_i)$. We have $N = n + \sum_i t_i$. The proof of Theorem 2.3 shows that the sequence $n_i - k_i$ is a strictly decreasing sequence of positive integers and that we have at most $n-1$ iterations. Therefore $N \leq n + \sum_{j=1}^{n-1} j = n(n+1)/2$. □

3. Geometry

We briefly recall some basics about Schubert cells, Schubert varieties and Richardson varieties (cf., for example, [F, §10.2]).

For $v_1, \dots, v_j \in \mathbb{C}^n$, we denote by $\langle v_1, \dots, v_j \rangle$ the $\mathbb{C}$-subspace of $\mathbb{C}^n$ that they span. We customarily denote full flags of subspaces of $\mathbb{C}^n$ by $V_\bullet = (V_1 \subset \cdots \subset V_{n-1})$, where $\dim V_i = i$. Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{C}^n$, and, for $j = 1, \dots, n-1$, we set

$$E_j = \langle e_1, \dots, e_j \rangle, \quad \text{and} \quad E_j^{\text{opp}} = \langle e_n, e_{n-1}, \dots, e_{n-j+1} \rangle.$$

Let $w \in S_n$. We recall the definition of the Schubert cell C_w:

$$C_w = \{V \in \mathcal{F}l_n \mid \dim(V_p \cap E_q) = k_{p,q} \quad 1 \le p, q \le n\},$$

where $k_{p,q} = \#\{i \le p \mid w(i) \le q\}$. Its closure in the flag variety is called the Schubert variety:

$$X_w = \{V \in \mathcal{F}l_n \mid \dim(V_p \cap E_q) \ge k_{p,q} \quad 1 \le p, q \le n\}. \tag{3.1}$$

For $v \in S_n$, the opposite Schubert cell and opposite Schubert variety are defined as

$$C^v = \left\{V \in \mathcal{F}l_n \mid \dim(V_p \cap E_q^{\text{opp}}) = h_{p,q} \quad 1 \le p, q \le n\right\},$$

$$X^v = \left\{V \in \mathcal{F}l_n \mid \dim(V_p \cap E_q^{\text{opp}}) \ge h_{p,q} \quad 1 \le p, q \le n\right\}, \tag{3.2}$$

where $h_{p,q} = \#\{i \le p \mid v(i) \ge n+1-q\}$.

For a pair of permutations $x \le y \in S_n$, we define the Richardson variety

$$X_y^x := X_y \cap X^x.$$

Theorem 3.1. *Let $v, w \in S_N$ with $v \le w$, and let $m \in \mathbb{Z}_{\ge 0}$ be such that $v(i) = w(i)$ for any $i \le m$. For $u \in \{v, w\}$ let $u' \in S_{N-m}$ be given by*

$$u'(j) = u(j+m) - \#\{k \le m \mid u(k) \le u(j+m)\}, \qquad j = 1, \dots, N-m.$$

Then there is an isomorphism of algebraic varieties $X_w^v \simeq X_{w'}^{v'}$.

Proof. By induction it suffices to consider the case $m = 1$. Let $t = v(1) = w(1)$. Let $\pi : \mathbb{C}^N \to \mathbb{C}^{N-1}$ be the projection forgetting the t-th coordinate.

Define an inclusion $\iota : \mathcal{F}l_{N-1} \to \mathcal{F}l_N$ by $\iota(V_\bullet) = (\pi^{-1}(V_i))_{i=0,\dots,N-2}$, where $V_0 = \{0\}$. Suppose that $V_\bullet \in X_{w'}^{v'}$. Then

$$\begin{aligned}
\dim(\pi^{-1}(V_{p-1} \cap E_q)) &= \begin{cases} \dim(V_{p-1} \cap E_q) & \text{if } q < t \\ \dim(V_{p-1} \cap E_{q-1}) + 1 & \text{if } q \ge t \end{cases} \\
&\ge \begin{cases} \#\{i \le p-1 \mid w'(i) \le q\} & \text{if} \quad q < t \\ \#\{i \le p-1 \mid w'(i) \le q-1\} + 1 & \text{if} \quad q \ge t \end{cases} \\
&= \#\{i \le p \mid w(i) \le q\}
\end{aligned}$$

and similarly we show that

$$\dim(\pi^{-1}(V_{p-1} \cap E_q^{\mathrm{opp}})) \geq \#\{i \leq p \mid v(i) \geq N+1-q\}.$$

Therefore $\iota(V^\bullet) \in X_w^v$, and we have an injective morphism of algebraic varieties $\iota' := \iota_{|X_{w'}^{v'}} : X_{w'}^{v'} \to X_w^v$.

Notice that, by (3.1), if $V_\bullet \in X_w^v$, then $\langle e_t \rangle \subseteq V_i$ for any $i = 1, \ldots, N-1$ and so we can define

$$\eta : X_w^v \to \mathcal{F}l_{N-1}, \quad V_\bullet = (V_i)_{i=1,\ldots,N-1} \mapsto (\pi(V_i))_{i=2,\ldots,N-1}.$$

Similar considerations as before on dimension bounds show that $\eta(X_w^v) \subseteq X_{w'}^{v'}$, and it is immediate to see that η and ι' are inverse to each other. □

Let T be the maximal torus of diagonal matrices in $GL_N(\mathbb{C})$. This acts on $\mathcal{F}l_N$ and since it stabilises the spaces E_j and E_j^{opp}, it also acts on X_w^v. The proof of Theorem 3.1 involves a quotient map from $\mathbb{C}^N$ to $\mathbb{C}^{N-m}$ which induces a T-action on $\mathbb{C}^{N-m}$ and hence on $\mathcal{F}l_{N-m}$. Then the isomorphism of Theorem 3.1 is T-equivariant.

Corollary 3.2. *With notation as in Theorem 3.1, the singularity type of X_w at v is isomorphic to the singularity type of $X_{w'}$ at v'.*

Proof. The isomorphism of Theorem 3.1 sends w to w', v to v' and is T-equivariant. Since the slice $C^v \cap X_w$ can be constructed in terms of the T-action on the Richardson variety, the two singularity types are the same. (cf. [KL2, Lemma 1.4]). □

Our main theorem from the introduction, Theorem 1.1, has now been proved. It is a combination of Theorem 2.3 and Corollary 3.2.

Remark 3.3. *The Bruhat intervals $[v, w]$ and $[v', w']$ are isomorphic. This can be checked algebraically or follows from our T-equivariant isomorphism of Richardson varieties.*

4. Some applications

4.1. **Associated varieties of highest weight modules.** Let ρ be half the sum of the positive roots of $\mathfrak{sl}_n$. For a permutation $w \in S_n$, we denote by L_w the simple $\mathfrak{sl}_n$-module of highest weight $-w(\rho) - \rho$. Denote by $\mathcal{D}$ the sheaf of algebraic linear differential operators on the flag variety $\mathcal{F}l_n$ and by $\mathcal{L}_w$ the $\mathcal{D}$-module corresponding to L_w under the Beilinson-Bernstein correspondence, that is the IC-extension of the constant local system of the cell C_y. Its characteristic variety $\mathtt{Ch}(\mathcal{L}_\mathtt{w})$ is a subvariety of the cotangent bundle $T^*\mathcal{F}l_n$ and the corresponding characteristic cycle is a $\mathbb{Z}_{\geq 0}$-linear combination of the classes of the (closures) of conormal bundles of the Schubert cells:

$$CC(\mathcal{L}_y) = \sum_{x \leq y} m_{x,y} [\overline{T_x^* \mathcal{F}l_n}].$$

Determining the numbers $m_{x,y}$ is a natural (but in general very hard) question. Notice that, as $m_{y,y} = 1$, the characteristic variety $\mathtt{Ch}(\mathcal{L}_{\mathtt{y}})$ is irreducible if and only if $m_{x,y} = 0$ for any $x \neq y$. The associated variety $V(L_w)$ can be obtained as the image of $\mathtt{Ch}(\mathcal{L}_{\mathtt{w}})$ under the moment map $\gamma : T^*\mathcal{F}l_n \to \mathfrak{g}^*$. It turns out that $V(L_w)$ is irreducible if and only if $m_{x,y} = 0$ for any $x \neq y$ such that x and y lie in the same two-sided Kazhdan-Lusztig cell (cf. [BB], [Jo]).

While several examples of reducible characteristic varieties have been known for long time, it was conjectured that the associated varieties to $\mathfrak{sl}_n$-highest weight modules were irreducible (cf. [BB, Conjecture 4.5] and [Jo, §10.2]). The first example of a reducible associated variety was exhibited by Williamson in [W]. This was found by a computer search motivated by constraints that must be satisfied in order for torsion to occur in the intersection cohomology of Schubert varieties.

Williamson's example ended up producing a singularity in the S_{12} flag variety where x and y lay in the same right cell. It was then shown that this singularity was smoothly equivalent to the one studied by Kashiwara and Saito which they computed the characteristic cycle for. Since the $m_{x,y}$'s only depend on the singularity type of the Schubert variety X_y at x, this yields a reducible associated variety. Our approach gives a simpler and systematic method for producing similar examples.

Remark 4.1. *By* [T, §3.1(A)]*, the irreducibility of an associated variety is equivalent to the coincidence of two bases (the* Goldie rank *basis and the* Springer basis*) of a complex irreducible S_n-representation, so that our algorithm also provides a method to determine examples of representations for which the two bases differ. A similar question is addressed in the following section, where we explain how to apply our main result to the comparison of bases of Specht modules for the Hecke algebra.*

4.2. **Bases of Specht modules.** Consider the Hecke algebra $H_n(q)$ of the symmetric group. Let $\{H_x\}_{x\in S_n}$ be the Kazhdan-Lusztig basis of $H_n(q)$. Let λ be a partition and Q a standard tableau of shape λ. Then the theory of cell modules shows that the set

$$\{H_x \mid Q(x) = Q\}$$

is a basis of the Specht module S^λ which does not depend on the choice of Q. We call this the KL basis.

Jensen [Je] studies the analogous situation for the p-Kazhdan-Lusztig basis $\{^pH_x\}_{x\in S_n}$ of $H_n(q)$ - defined in terms of parity sheaves on the flag variety. Here p is a prime. One defines left, right and 2-sided p-cells analogously to the case of the Kazhdan-Lusztig basis. Jensen [Je, Theorem 4.33] shows that these p-cells are the same as the Kazhdan-Lusztig cells. Furthermore, in [Je, Corollary 4.39], he shows that

$$\{^pH_x \mid Q(x) = Q\}$$

is a basis of the Specht module S^λ which does not depend on the choice of Q. We call this the p-KL basis.

Define the transition matrix $({}^p m_{xy})_{x,y \in S_n}$ by

$$ {}^p H_x = \sum_y {}^p m_{xy} H_y. $$

If we can find x and y with $x \neq y$, $Q(x) = Q(y)$ and ${}^p m_{xy} \neq 0$, then the KL and p-KL bases of the corresponding Specht module will disagree. Since ${}^p m_{xy}$ is an invariant of the singularity type of X_x at y, our main theorem is designed to provide such examples, whenever we can find any $u \neq v$ with ${}^p m_{uv} \neq 0$. For instance, in Example 2.4, ${}^2 m_{x,y} = 1$, and $P(u) = P(v)$ have shape $\lambda = (4,4,2,2)$, hence the KL basis of $S^{(4,4,2,2)}$ disagrees with the 2-KL basis.

For instance, transferring the torsion explosion examples from [WKM] and applying Proposition 2.6, we get infinitely many examples where these bases differ with $n < A(\log p)^2$ for some constant A (we have made no effort to optimise this bound).

By transferring the examples of non-perverse parity sheaves from [Mc], we see that the change of basis matrix within a Specht module can contain polynomials in q of arbitrarily large degree.

The results of [N], together with the positivity property of the p-Kazhdan-Lusztig basis under multiplication, imply that whenever these bases disagree, the corresponding W-graph for the p-KL basis is not bipartite. An explicit example of a non-bipartite W-graph for $p = 2$ in type C_3 was previously exhibited in [Je].

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