CLUSTER MONOMIALS ARE DUAL CANONICAL

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Abstract. Kang, Kashiwara, Kim and Oh have proved that cluster monomials lie in the dual canonical basis, under a symmetric type assumiption. This involves constructing a monoidal categorification of a quantum cluster algebra using representations of KLR algebras. We use a folding technique to generalise their results to all Lie types.

Contents

1. Introduction

Let G be a Kac-Moody group, w an element of its Weyl group. Then asssociated to w and G there is a unipotent group $N(w)$. The coordinate ring $\mathbb{C}[N(w)]$ has a natural structure of a cluster algebra [GLS11]. There is a quantum version of this story as well [GLS13a, GY] where the quantised coordinate ring $A_q(\mathfrak{n}(w))$ has a quantum cluster algebra structure. In this paper, we work with the quantum version. However, for this introduction, we shall continue with the classical story.

Kashiwara [Kas91] and Lusztig [Lus90, Lus91] defined a remarkable basis of the enveloping algebra $U(\mathfrak{n})$, called the lower global base or the canonical basis. It's dual, the dual canonical

Date: October 28, 2017.

basis, is a basis of $\mathbb{C}[N]$ and is of concern to us. This dual canonical basis further induces a basis of the cluster algebra $\mathbb{C}[N(w)]$.

It was a long-standing conjecture that the cluster monomials lie in the dual canonical basis. This was recently proved whenever G is symmetric by Kang, Kashiwara, Kim and Oh [KKKOa, KKKOb]. Their proof used the categorification of $A_q(\mathfrak{n}(w))$ by categories of modules over Khovanov-Lauda-Rouquier algebras (henceforth called KLR algebras). Briefly it proceeds by finding a monoidal categorification of the cluster algebra structure inside these categories of KLR modules.

In this paper we follow in the footsteps of Lusztig's approach to the canonical basis in non-symmetric types using the technique of folding by an automorphism of the Dynkin diagram. The corresponding theory of folding KLR algebras was recently developed in [McNa]. Inside these folded categories, we are able to fold the monoidal categorification of [KKKOa, KKKOb] to deduce the fact that the cluster monomials lie in the dual canonical basis, and more generally, they lie in the dual p-canonical basis for all p.

Our main theorem that we prove in this paper is the following:

Theorem 1.1. *The algebra $\mathcal{A}_q(\mathfrak{n}(w))$ has the structure of a (explicitly defined, independent of p) quantum cluster algebra in which every cluster monomial lies in the dual p-canonical basis.*

We now briefly discuss the contents of this paper. We begin with an overview of the theory of generalised minors and bases of canonical type (which are closely related to perfect bases).

We then summarise the necessary background results about folded KLR algebras. These results are all proved in [McNa].

Finally we discuss the work of Kang, Kashiwara, Kim and Oh, and show how to incorporate the diagram automorphism into their story. This is where we prove the main result of the paper, Theorem **??**.

The results of this paper prove that certain modular decomposition numbers for KLR algebras are trivial. In particular, they prove that the reduction modulo p of any irreducible module which corresponds to a cluster monomial remains irreducible. Such results also have some geometric consequences, implying the non-existence of torsion in the stalks and costalks of the intersection cohomology of certain (Lusztig) quiver varieties. For example, by [Wil14, Theorem 3.7], there is no such torsion in A_4, as in this case $\mathcal{A}_q(\mathfrak{n})$ is a finite type cluster algebra.

We thank B. Leclerc for useful discussions about the theory of cluster algebras.

2. The quantum group

Let $\mathfrak{g}$ be a symmetrisable Kac-Moody Lie algebra and I and a set indexing the simple roots. To each $i \in I$ is an associated integer d_i which is the entry of the symmetrising matrix.

We work over the ring $\mathbb{Z}[q, q^{-1}]$ or its fraction field $\mathbb{Q}(q)$. For each $i \in I$, let $q_i = q^{d_i}$. The quantum integer $[n]_i$ is $(q_i^n - q_i^{-n})/(q_i - q_i^{-1})$. The quantum factorial $[n]_i!$ is the product $[n]_i[n-1]_i \cdots [1]_i$.

Let $U_q(\mathfrak{g})$ be the corresponding quantised enveloping algebra. Write $U_q(\mathfrak{n})$ for the upper-triangular part and let $\mathcal{A}_q(\mathfrak{n})$ be its graded dual. Let $\{\theta_i\}_{i\in I}$ denote the usual generating set of $U_q(\mathfrak{n})$ as an algebra, and use θ_i^* to denote the dual generating set of $\mathcal{A}_q(\mathfrak{n})$. We work with the $\mathbb{Z}[q,q^{-1}]$-form of $U_q(\mathfrak{n})$ generated as an algebra by the divided powers $\theta_i^{(n)} := \theta_i^n/[n]_i!$ and the corresponding $\mathbb{Z}[q^{\pm 1}]$-form of $A_q(\mathfrak{n})$ which is its graded integral dual. When writing $U_q(\mathfrak{n})$ or $\mathcal{A}_q(\mathfrak{n})$, we will always refer to this integral form.

The bar involution on $U_q(\mathfrak{n})$ is the automorphism fixing the Chevalley generators and sending q to q^{-1}. This involution induces a bar involution on the dual $\mathcal{A}_q(\mathfrak{n})$. For homogenous elements a and b of degrees α and β in $\mathcal{A}_q(\mathfrak{n})$, we have the formula [Lec04, Proposition 1]

$$\overline{ab} = q^{\alpha\cdot\beta}\bar{b}\bar{a}. \tag{2.1}$$ {barproduct}

Let P^+ denote the set of dominant weights, identified with $\mathbb{N}I$, and let P be the weight lattice. If $i \in I$, let ω_i be the corresponding fundamental weight. For each $\lambda \in P^+$, we denote by $V(\lambda)$ the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ. There is a partial order on P where $\lambda \geq \mu$ if $\lambda - \mu$ is a sum of positive roots.

The algebra $A_q(\mathfrak{n})$ is graded by $\mathcal{P}^+$, we write $A_q(\mathfrak{n})_\lambda$ for the λ'th graded piece.

Write $E_i (= \theta_i)$ and F_i for the usual generators of $U_q(\mathfrak{g})$. Let ϕ be the involutive anti-automorphism of $U_q(\mathfrak{g})$ sending E_i to F_i. Again, we write $F_i^{(c)} := F_i^c/[c]_i!$ for the divided power.

Let $(\cdot,\cdot)$ denote the q-Shapovalov form on $V(\lambda)$. This is a nondegenerate bilinear form such that

$$(xv, w) = (v, \phi(x)w)$$

for all $x \in U_q(\mathfrak{g})$ and $v, w \in V(\lambda)$. We normalise the q-Shapovalov form so that $(v_\lambda, v_\lambda) = 1$, where v_λ is a chosen highest weight vector.

We now define a weight vector v_μ for all extremal weights μ in $V(\lambda)$. Such an extremal weight μ is of the form $w\lambda$ for some $w \in W$. Let $w = s_{i_1}\cdots s_{i_n}$ be a reduced decomposition of w. Then we define

$$v_\mu = F_{i_1}^{(c_1)}\cdots F_{i_n}^{(c_n)} v_\lambda$$

where the integers c_k are defined by $c_k = (\alpha_{i_k}, s_{i_{k+1}}\cdots s_{i_n}\lambda)$. The element v_μ does not depend on the choice of w or the choice of reduced decomposition.

The q-Shapovalov form satisfies $(v_\mu, v_\mu) = 1$ for all extremal weight vectors v_μ. This is proved by a rank one computation.

The coproduct is denoted $r_{\nu_1,\dots,\nu_n} : \mathcal{A}_q(\mathfrak{n}) \longrightarrow \mathcal{A}_q(\mathfrak{n}) \otimes \cdots \mathcal{A}_q(\mathfrak{n})$.

3. Bases of Canonical type

We need the notion of a basis of dual canonical type. This is a strengthening of the notion of a perfect basis of $A_q(\mathfrak{n})$.

For $i \in I$ and $p \in \mathbb{N}$, define ${}_{i^p}r$ and r_{i^p} to be the linear operators on $\mathcal{A}_q(\mathfrak{n})$ which are the adjoints of left and right multiplication by $\theta_i^{(p)}$ respectively.

Let σ be the antiautomorphism which fixes the Chevalley generators. The bar involution is the automorphism fixing the Chevalley generators and sending q to q^{-1}.

{def:can}

Definition 3.1. *A basis* $\mathbf{B}$ *of* $U_q(\mathfrak{n})$ *is said to be of canonical type if it satisfies the conditions (1)–(6) below:*

{it:BOCTa}

(1) *The elements of* $\mathbf{B}$ *are weight vectors.*

{it:BOCTb}

{it:BOCTc} (2) $1 \in \mathbf{B}$.

{it:BOCTd} (3) *Each right ideal* $(\theta_i^p \mathbf{U}_q(\mathfrak{n}) \otimes \mathbb{Q}(q)) \cap U_q(\mathfrak{n})$ *is spanned by a subset of* $\mathbf{B}$.

(4) *In the bases induced by* $\mathbf{B}$*, the left multiplication by* $\theta_i^{(p)}$ *from* $U_q(\mathfrak{n})/\theta_i U_q(\mathfrak{n})$ *onto* $\theta_i^p U_q(\mathfrak{n})/\theta_i^{p+1} U_q(\mathfrak{n})$ *is given by a permutation matrix.*

{it:BOCTe} (5) $\mathbf{B}$ *is stable by* σ.

{it:BOTf} (6) $\mathbf{B}$ *is stable under the bar involution.*

dualcantype}

Definition 3.2. *A basis* $\mathbf{B}^*$ *of* $\mathcal{A}_q(\mathfrak{n})$ *is said to be of dual canonical type if it satisfies the conditions (1)–(6) below:*

{it:BOdCTa} (1) *The elements of* $\mathbf{B}^*$ *are weight vectors.*

{it:BOdCTb} (2) $1 \in \mathbf{B}^*$.

{it:BOdCTc} (3) *Each* $\ker(r_{i^p})$ *is spanned by a subset of* $\mathbf{B}^*$.

{it:BOdCTd} (4) *In the bases induced by* $\mathbf{B}^*$*, the map*

$$r_{i^p} : \ker(r_{i^{p+1}})/\ker(r_{i^p}) \longrightarrow \ker(r_i)$$

is given by a permutation matrix.

{it:BOdCTe} (5) $\mathbf{B}^*$ *is stable by* σ.

{it:BOdCTf} (6) $\mathbf{B}^*$ *is stable under the bar involution.*

The dual basis to a basis of canonical type is of dual canonical type and vice versa.

It is proved in [BK07, §5] that each basis of dual canonical type induces the crystal $B(\infty)$.

{basisofvla}

Theorem 3.3. [Bau, McNa] *Let* $\lambda \in P^+$*. Let* $V(\lambda)$ *be a highest weight module of* $U_q(\mathfrak{g})$ *with highest weight* λ *and let* v_λ *be a highest weight vector. Let* $\mathbf{B}$ *be a basis of canonical type. Then the set*

$$\{bv_\lambda \mid b \in \mathbf{B}, bv_\lambda \neq 0\}$$

is a basis of $V(\lambda)$.

The algebra $U_q(\mathfrak{n})$ is the quotient of the free algebra $'\mathbf{f}$ on the generators $\{\theta_i\}_{i\in I}$. The shuffle algebra Ш is defined as the graded dual of $'\mathbf{f}$, where the basis of words $[i_1, \ldots, i_n]$ in Ш is dual to the basis of monomials in $'\mathbf{f}$. Then $A_q(\mathfrak{n})$ is a subalgebra of Ш and we write ι for the inclusion.

Let $i_1, i_2, \ldots$ be a sequence of elements of I such that each element of I appears infinitely often. A word is said to be extremal for an element $x \in \mathcal{A}_q(\mathfrak{n})$ if it is minmial for the relevant lexicographical order amongst all words which appear in $\iota(x)$ with nonzero coefficient.

tremalwords}

Lemma 3.4. *Let* $\mathbf{B}^*$ *be a basis of dual canonical type and let* $b^* \in \mathbf{B}^*$*. Let* $\mathbf{i} = i_1^{a_1} i_2^{a_2} \cdots$ *be an extremal word for* b^**. Then* $\mathbf{i}$ *appears in* $\iota(b^*)$ *with coefficient*

$$[a_1]_{i_1}![a_2]_{i_2}! \cdots .$$

Proof. Consider ${}_{i_1^{a_1}}r(b^*)$. By Condition (4) in definition 3.2, it lies in $\mathbf{B}^*$. The word $i_2^{a_2} i_3^{a_3} \ldots$ is extremal for ${}_{i_1^{a_1}}r(b^*)$ and by induction, we may assume that it appears with coefficient $[a_2]_{i_2}! \cdots$ in $\iota({}_{i_1^{a_1}}r(b^*))$. This implies that $\mathbf{i}$ appears in $\iota(b^*)$ with the desired coefficient, namely $[a_1]![a_2]! \cdots$. $\square$

{product} **Lemma 3.5.** [Kle, §2.8] *Let $\mathbf{B}^*$ be a basis of dual canonical type and let x and y be two elements of $\mathbf{B}^*$. Expand their product in the basis $\mathbf{B}^*$:*

{exp}
$$xy = \sum_{b^* \in \mathbf{B}^*} c_{b^*} b^*. \tag{3.1}$$

Then there exists b^* such that c_{b^*} is a power of q.

Proof. We have an extremal word $i_1^{a_1} i_2^{a_2} \ldots$ for x and an extremal word $i_1^{b_1} i_2^{b_2} \cdots$ for y. Then the word $i_1^{a_1+b_1} i_2^{a_2+b_2} \ldots$ is extremal for xy and appears with multiplicity

$$[a_1 + b_1]![a_2 + b_2]! \cdots$$

by Lemma 3.4. Apply $\cdots r_{i_2^{a_2+b_2}} r_{i_1^{a_1+b_1}}$ to (3.1). The left hand side is a power of q, while by extremality, each term in the right hand side is c_{b^*} times an element of $\mathbf{B}$ or zero. The element $1 \in \mathbf{B}^*$ can only appear once, so there must be a term where c_{b^*} is a power of q. □

For any weight basis $\mathbf{B}$, let $\mathbf{B}_\nu$ be the elements in $\mathbf{B}$ of weight ν. {gmlemma}

Lemma 3.6. *Let $\mathbf{B}$ be a basis of canonical type. Let $\lambda \in P^+$ and let $\mu \leq \eta$ be two elements of $W\lambda$. Then there is exactly one choice of $b \in \mathbf{B}_{\eta-\mu}$ such that $bv_\mu \neq 0$. For this choice of b we have $bv_\mu = v_\eta$.*

Proof. Since the q-Shapovalov form is nondegenerate and the η-weight space of $V(\lambda)$ is one-dimensional, $bv_\mu \neq 0$ if and only if $(bv_\mu, v_\eta) \neq 0$.

We concentrate on the first statement and proceed by induction on η. First consider the base case when $\eta = \lambda$. In this case we have $(bv_\mu, v_\eta) = (v_\mu, \sigma(b)v_\eta)$. The invariance of $\mathbf{B}$ under σ implies that $\phi(b) \in \mathbf{B}$ if and only if $b \in \mathbf{B}$. By Theorem 3.3, there is a unique choice of $\phi(b) \in \mathbf{B}$ making this pairing nonzero, hence a unique choice of $b \in \mathbf{B}$.

Now consider the case of $\eta \neq \lambda$. Then there exists i such that $\eta < s_i\eta$. Then $v_\eta = F_i^{(c)} v_{s_i\eta}$ for some integer c. We compute

$$(bv_\mu, v_\eta) = (bv_\mu, F_i^{(c)} v_{s_i\eta}) = (E_i^{(c)} bv_\mu, v_{s_i\eta}).$$

If $b \in E_i U_q(\mathfrak{n})$ then $E_i^{(c)} bv_\mu$ factors through the weight space $V(\lambda)_{\eta-\alpha_i}$ which is zero. If $b \notin E_i U_q(\mathfrak{n})$, then there exists a unique $b' = \tilde{E}_i^c b \in \mathbf{B}$ such that $E_i^{(c)} b - b' \in E_i^{c+1} U_q(\mathfrak{n})$. By a similar argument, $E_i^{(c)} bv_\eta = b'v_\eta$.

By the inductive hypothesis, there exists a unique choice of $b' \in \mathbf{B}$ such that $(b'v_\mu, v_{s_i\eta}) = 0$. As $\tilde{e}_i$ is injective, there is thus at most one choice of b such that $(bv_\mu, v_\eta) \neq 0$. The existence of such a b is obvious as the condition $\mu \leq \eta$ means that it is easy to write down a product x of Chevalley generators in $U_q(\mathfrak{n})_{\eta-\mu}$ such that $xv_\mu \neq 0$.

At this point we have proved the existence of exactly one choice of $b \in \mathbf{B}_{\eta-\mu}$ such that $bv_\mu \neq 0$. We now aim to show that for this choice of b, we have $bv_\mu = v_\eta$. We induct on η, the base case where $\eta = \mu$ being trivial.

Suppose then that $\eta \neq \mu$. Then there exists i such that $\mu \leq s_i\eta < \eta$. Then by the inductive hypothesis, there exists $b' \in \mathbf{B}$ such that $b'v_\mu = v_{s_i\eta}$. Then $E_i^{(c)} b'v_\mu = v_\eta$ and we do a similar argument to show that $(\tilde{e}_i^c b')v_\mu = v_\eta$ where $\tilde{e}_i$ is the crystal operator, so by uniqueness of b, we're done. □

4. GENERALISED MINORS

Definition 4.1. *Let $\lambda \in P^+$ and $\mu, \eta \in W\lambda$. The generalised minor $D(\mu, \eta) \in \mathcal{A}_q(\mathfrak{n})$ is defined by*

$$D(\mu, \eta)(x) = (xv_\mu, v_\eta)$$

for all $x \in U_q(\mathfrak{n})$.

The agreement between this definition and the definition of [KKKOb] is discussed in [GLS11, §5].

{genminors}

Theorem 4.2. *Let $\mathbf{B}^*$ be a basis of dual canonical type. All generalised minors $D(\mu, \eta)$ with $\mu \leq \eta$ lie in $\mathbf{B}^*$.*

Proof. Suppose $b \in \mathbf{B}$ where $\mathbf{B}$ is the basis dual to $\mathbf{B}^*$. Then $D(\mu, \eta)(b) = (bv_\mu, v_\eta)$. By Lemma 3.6, there is a unique choice of $b \in \mathbf{B}$ such that this is nonzero, and for this particular choice of b, $bv_\mu = v_\eta$. Therefore $D(\mu, \eta) \in \mathbf{B}^*$. □

{restrictgm}

Lemma 4.3. *Let $\lambda \in P^+$ and suppose that $\mu_1 < \mu_2 < \cdots < \mu_{n+1}$ are weights in $W\lambda$. Then*

$$r_{\mu_1-\mu_2,\ldots,\mu_n-\mu_{n+1}}(D(\mu_1, \mu_{n+1})) = D(\mu_1, \mu_2) \otimes \cdots \otimes D(\mu_n, \mu_{n+1}).$$

Proof. Let $\mathbf{B}$ be a basis of canonical type. Suppose that $b_1, \ldots, b_n \in \mathbf{B}$ are such that

$$(r_{\mu_1-\mu_2,\ldots,\mu_n-\mu_{n+1}}(D(\mu_1, \mu_{n+1})), b_1 \otimes \cdots \otimes b_n) \neq 0.$$

Then $(D(\mu_1, \mu_{n+1}), b_1 \cdots b_n) \neq 0$. By the definition of the generalised minor, this implies that

$$(v_{\mu_1}, b_1 \cdots b_n v_{\mu_{n+1}}) \neq 0.$$

By repeated application of Lemma 3.6, there is a unique choice of $b_1, \ldots, b_n \in \mathbf{B}$ such that this pairing is nonzero, and for this choice of $b_1, \ldots, b_n$, the value of the pairing is 1.

Furthermore, by looking at the degrees involved, we see that the identity

$$b_i v_{\mu_{i+1}} = v_{\mu_i}$$

holds for each i.

Therefore we have identified $r_{\mu_1-\mu_2,\ldots,\mu_n-\mu_{n+1}}(D(\mu_1, \mu_{n+1}))$ as a tensor product of elements of $\mathbf{B}^*$. Furthermore the proof of Lemma 3.6 identifies each factor as $D(\mu_i, \mu_{i+1})$. □

{squaregm}

Lemma 4.4. *Let λ be a dominant weight and $\mu, \zeta \in W\lambda$. Then*

$$D(\mu, \zeta)^2 = q^{(\mu-\zeta,\mu-\zeta)/2} D(2\mu, 2\zeta).$$

Proof. The submodule of $V(\lambda) \otimes V(\lambda)$ generated by $v_\lambda \otimes v_\lambda$ is isomorphic to $V(2\lambda)$. Now let us specialise to $q = 1$. Then we can define a bilinear form on $V(\lambda) \otimes V(\lambda)$ by setting

$$(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2, y_2)$$

and extending by linearity, where $(\cdot, \cdot)$ is the Shapovalov form. When restricted to $V(2\lambda)$ inside $V(\lambda) \otimes V(\lambda)$ this form satisfies

$$(xv, w) = (v, \phi(x)w)$$

for all $x \in \mathfrak{g}$, hence is the Shapovalov form on $V(2\lambda)$.

Note that $v_\mu \otimes v_\mu$ and $v_\zeta \otimes v_\zeta$ are normalised extremal weight vectors in $V(2\lambda)$. Thus at $q = 1$, we have

$$\begin{aligned} D(2\mu, 2\zeta)(x) &= (x(v_\mu \otimes v_\mu), v_\zeta \otimes v_\zeta) \\ &= (D(\mu,\zeta) \otimes D(\mu,\zeta))(\Delta(x)) \\ &= D(\mu,\zeta)^2(x). \end{aligned}$$

We have thus proved this lemma when q is specialised to 1. Now pick a basis $\mathbf{B}^*$ of dual canonical type such that the structure constants for multiplication all lie in $\mathbb{N}[q, q^{-1}]$. For example, by [McNa, Theorem 12.7], we could take a basis coming from simple representations of KLR algebras. From Theorem 4.2, $D(\mu,\zeta)$ and $D(2\mu, 2\zeta)$ both lie in $\mathbf{B}^*$. So by considering the expansion of $D(\mu,\zeta)^2$ in the basis $\mathbf{B}^*$ and comparing with what we already know about the behaviour at $q = 1$, the only option is that $D(\mu,\zeta)^2 = q^N D(2\mu, 2\zeta)$ for some integer N. We can identify the integer N from the identity (2.1). □

5. KLR algebras with automorphism

Let Q be a quiver with vertex set I and let a be a finite order automorphism of Q. We assume that there are no arrows between any pair of vertices in the same a-orbit. Let n be the order of a.

To such a quiver with automorphism, let J be the set of a-orbits on I. Define $\cdot : J \times J \longrightarrow \mathbb{Z}$ by $j \cdot j = 2|j|$ and for $j \neq k$, $-j \cdot k$ is equal to the total number of edges in Q between an element of j and an element of k. This is a symmetrisable Cartan datum expressed using the formulation in [Lus93]. It is known that every symmetrisable Cartan datum arises from such a construction. The relationship between (Q, a) and the Kac-Moody Lie algebra $\mathfrak{g}$ we work with is that $\mathfrak{g}$ is the Kac-Moody Lie algebra with Cartan datum $(J, \cdot)$.

Define, for any $\nu \in \mathbb{N}I$,

$$\mathrm{Seq}\,(\nu) = \{\mathbf{i} = (\mathbf{i}_1, \ldots, \mathbf{i}_{|\nu|}) \in I^{|\nu|} \mid \sum_{j=1}^{|\nu|} \mathbf{i}_j = \nu\}.$$

This is acted upon by the symmetric group $S_{|\nu|}$ in which the adjacent transposition $(i, i+1)$ is denoted s_i.

Define the polynomials $Q_{i,j}(u, v)$ for $i, j \in I$ by

$$Q_{i,j}(u,v) = \begin{cases} \prod_{i\to j}(u-v)\prod_{j\to i}(v-u) & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

where the products are over the sets of edges in Q from i to j and from j to i, respectively.

Let k be an algebraically closed field whose characteristic does not divide n.

Definition 5.1. *The KLR algebra $R(\nu)$ is the associative k-algebra generated by elements $e_{\mathbf{i}}$, y_j, τ_k with $\mathbf{i} \in$ Seq(ν), $1 \le j \le |\nu|$ and $1 \le k < |\nu|$ subject to the relations*

$$\begin{aligned}
&e_{\mathbf{i}} e_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}} e_{\mathbf{i}}, \quad \sum_{\mathbf{i} \in \mathrm{Seq}(\nu)} e_{\mathbf{i}} = 1, \\
&y_k y_l = y_l y_k, \quad y_k e_{\mathbf{i}} = e_{\mathbf{i}} y_k, \\
&\tau_l e_{\mathbf{i}} = e_{s_l \mathbf{i}} \tau_l, \quad \tau_k \phi_l = \tau_l \phi_k \quad \text{if } |k-l| > 1, \\
&\tau_k^2 e_{\mathbf{i}} = Q_{\mathbf{i}_k, \mathbf{i}_{k+1}}(y_k, y_{k+1}) e_{\mathbf{i}}, \\
&(\tau_k y_l - y_{s_k(l)} \tau_k) e_{\mathbf{i}} = \begin{cases} -e_{\mathbf{i}} & \text{if } l = k, \mathbf{i}_k = \mathbf{i}_{k+1}, \\ e_{\mathbf{i}} & \text{if } l = k+1, \mathbf{i}_k = \mathbf{i}_{k+1}, \\ 0 & \text{otherwise}, \end{cases} \\
&(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e_{\mathbf{i}} \\
&= \begin{cases} \dfrac{Q_{\mathbf{i}_k, \mathbf{i}_{k+1}}(y_k, y_{k+1}) - Q_{\mathbf{i}_k, \mathbf{i}_{k+1}}(y_{k+2}, y_{k+1})}{y_k - y_{k+2}} e_{\mathbf{i}} & \text{if } \mathbf{i}_k = \mathbf{i}_{k+2}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{5.1}$$ {eq:KLR}

We remark that we have not used the most generic choice of polynomials $Q_{ij}(u,v)$ as in [Rou] to define these algebras. However it is important to us that we do use this choice, which implies that these algebras are isomorphic to certain Ext algebras [VV11, Rou12, Mak15] on the moduli stack of representations of the quiver Q. We rely on some results from [KKKOb] which require this geometric interpretation of these algebras. This assumption also implies that the algebras $R(\nu)$ *symmetric* in the sense of [KKK], giving us access to the theory of R-matrices for KLR algebras.

The algebras $R(\nu)$ are $\mathbb{Z}$-graded by setting $e_{\mathbf{i}}$ to have degree zero, y_i to have degree 2 and $\tau_i e_{\mathbf{i}}$ to have degree $\mathbf{i}_i \cdot \mathbf{i}_{i+1}$. All $R(\nu)$-modules which we consider in this paper will be graded left modules.

The automorphism a of Q induces isomorphisms $R(\nu) \cong R(a\nu)$ for all ν. In particular, when $a\nu = \nu$, it induces an automorphism of the algebra $R(\nu)$, which we will also denote by a.

Consider ν such that $a\nu = \nu$. We consider the category $\mathcal{C}_\nu$ of pairs (M, σ) where M is a representation of $R(\nu)$ and $\sigma : a^* M \longrightarrow M$ is an isomorphism such that

{atonis1} $$\sigma \circ a^* \sigma \circ \cdots \circ (a^*)^{n-1} \sigma = \mathrm{id}_M. \tag{5.2}$$

A morphism from (M, σ) to (N, τ) is a $R(\nu)$-module map $f : M \longrightarrow N$ such that the following diagram commutes:

$$\begin{array}{ccc}
a^* M & \xrightarrow{a^* f} & a^* N \\
\downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\
M & \xrightarrow{f} & N
\end{array}$$

Let $\mathcal{L}_\nu$ be the full subcategory of $\mathcal{C}_\nu$ whose objects are pairs (M, σ) where M is finite dimensional. In [McNa] it is shown that $\mathcal{C}_\nu$ and $\mathcal{L}_\nu$ are abelian categories.

6. The Grothendieck Group Construction

Let $\mathbb{Z}[\zeta_n]$ denote the ring of cyclotomic integers where ζ_n is a primitive n-th root of unity and fix a ring homomorphism $\mathbb{Z}[\zeta_n] \to k$.

An object (A, σ) of $\mathcal{C}_\nu$ is said to be *traceless* if there is a representation M of $R(\nu)$, an integer $t \geq 2$ dividing n such that $(a^*)^t M \cong M$, and an isomorphism

$$A \cong M \oplus a^* M \oplus \cdots \oplus (a^*)^{t-1} M$$

under which σ corresponds to an isomorphism carrying the summand $(a^*)^j M$ onto $(a^*)^j M$ for $1 \leq j < t$ and the summand $(a^*)^t M$ onto M.

The group $K(\mathcal{L}_\nu)$ is defined to be the $\mathbb{Z}[\zeta_n]$-module generated by symbols $[(M, \sigma)]$ where (M, σ) is an object of $\mathcal{L}_\nu$, subject to the relations

$$\begin{aligned} [X] &= [X'] + [X''] \qquad \text{if } 0 \to X' \to X \to X'' \to 0 \text{ is exact} \\ [(M, \zeta_n \sigma)] &= \zeta_n [(M, \sigma)] \\ [X] &= 0 \qquad\qquad\qquad \text{if } X \text{ is traceless} \end{aligned}$$

There is also a version of this construction for the category of finitely generated projective modules. It plays an important role in [McNa] but is not needed in this paper.

There is an action of q on $K(\mathcal{L}_\nu)$ by shifting the grading. Thus $K(\mathcal{L}_\nu)$ is naturally a module over the ring $\mathbb{Z}[\zeta_n, q, q^{-1}]$.

7. Induction, Restriction and Duality

Given (M, σ) and (N, τ) in $\mathcal{C}_\lambda$ and $\mathcal{C}_\mu$ respectively, we can form the induced module

$$M \circ N := R(\lambda + \mu) \bigotimes_{R(\lambda) \otimes R(\mu)} M \otimes N.$$

The isomorphisms σ and τ induce an isomorphism $a^* M \circ a^* N \to M \circ N$. When precomposed with the natural isomorphism $a^*(M \circ N) \cong a^* M \circ a^* N$, we obtain an isomorphism

$$\sigma \circ \tau : a^*(M \circ N) \to M \circ N.$$

The object

$$(M, \sigma) \circ (N, \tau) := (M \circ N, \sigma \circ \tau)$$

is an element of $\mathcal{C}_{\lambda+\mu}$.

This induction functor induces a product structure on the direct sum of Grothendieck groups

$$\bigoplus_{\nu \in \mathbb{N}J} K(\mathcal{L}_\nu).$$

For $\lambda, \mu \in \mathbb{N}J$, let $e_{\lambda\mu}$ be the image of the identity under the inclusion $R(\lambda) \otimes R(\mu) \to R(\lambda + \mu)$. Given a $R(\lambda + \mu)$-module M, its restriction is defined by

$$\operatorname{Res}_{\lambda,\mu} M := e_{\lambda\mu} M.$$

It is a $R(\lambda) \otimes R(\mu)$-module.

Since $e_{\lambda\mu}$ is invaraint under a, there is a canonical isomorphism $a^*(\operatorname{Res} M) \cong \operatorname{Res}(a^*M)$. Thus we obtain a restriction functor from $\mathcal{C}_{\lambda+\mu}$ to $\mathcal{C}_{\lambda\sqcup\mu}$. This restriction functor induces a coproduct structure on the same direct sum of Grothendieck groups

$$\bigoplus_{\nu\in\mathbb{N}J} K(\mathcal{L}_\nu),$$

the details of which can be found in [McNa].

Let ψ be the antiautomorphism of $R(\nu)$ which sends each of the generators $e_{\mathbf{i}}$, y_j and τ_k to themselves.

Let M be a finite dimensional $R(\nu)$-module. Then its dual $\mathbb{D}(M) := \operatorname{Hom}_k(M, k)$ is also an $R(\nu)$-module by

$$r(\lambda)(m) = \lambda(\psi(r)m)$$

for all $r \in R(\nu)$, $\lambda \in \mathbb{D}(M)$ and $m \in M$. This extends to a contravariant autoequivalence of $\mathcal{L}_\nu$ which we also denote $\mathbb{D}$, where $\mathbb{D}(L,\sigma) = (\mathbb{D}L, (\mathbb{D}\sigma)^{-1})$.

An object (M,σ) of $\mathcal{L}_\nu$ is said to be self-dual if there is an isomorphism

$$(M,\sigma) \cong (\mathbb{D}M, (\mathbb{D}\sigma)^{-1}).$$

Let $\mathbf{k}^*$ be the $\mathbb{Z}[q,q^{-1}]$-submodule of $\bigoplus_{\nu\in\mathbb{N}J} K(\mathcal{L}_\nu)$ spanned by the self-dual simple modules. For each $j \in J$, there is a canonical self-dual simple object of $\mathcal{L}_j$ which we denote by $L(j)$. It is unique if n is odd and unique up to scaling σ by ± 1 if n is even. A canonical choice is made in [McNa, §7].

thm:folding} **Theorem 7.1.** [McNa, Theorem 6.2] *There is a $\mathbb{Z}[q,q^{-1}]$-linear grading preserving isomorphism $\gamma^* : \mathbf{k}^* \longrightarrow \mathcal{A}_q(\mathfrak{n})$ such that*

1. $\gamma^*([L(j)]) = \theta_j^*$ *for all* $j \in J$.
2. *Under the isomorphism γ^*, the multiplication $\mathcal{A}_q(\mathfrak{n})_\lambda \otimes \mathcal{A}_q(\mathfrak{n})_\mu \to \mathcal{A}_q(\mathfrak{n})_{\lambda+\mu}$ corresponds to the product on $\mathbf{k}^*$ induced by $\operatorname{Ind}_{\lambda,\mu}$.*
3. *Under the the isomorphism γ^*, the comultiplication $\mathcal{A}_q(\mathfrak{n})_{\lambda+\mu} \to \mathcal{A}_q(\mathfrak{n})_\lambda \otimes \mathcal{A}_q(\mathfrak{n})_\mu$ corresponds to the coproduct on $\mathbf{k}^*$ induced by $\operatorname{Res}_{\lambda,\mu}$.*
4. *Under the isomorphism γ^*, the bar involution on $\mathcal{A}_q(\mathfrak{n})$ corresponds to the anti-linear antiautomorphism on $\mathbf{k}^*$ induced by the duality $\mathbb{D}$.*

It is shown in [McNa] that classes of self-dual simple objects give a basis of $\mathbf{k}^*$, which by Theorem 7.1 is transported to a basis of $\mathcal{A}_q(\mathfrak{n})$. We denote this basis by $\mathbf{B}^*$. It is also shown that $\mathbf{B}^*$ is a basis of dual canonical type. We call $\mathbf{B}^*$ the dual p-canonical basis, where p is the characteristic of k. When $p = 0$, $\mathbf{B}^*$ is the usual dual canonical basis, also known as the upper global basis.

8. Cuspidal modules

For now we work with the unfolded Dynkin diagram obtained by forgetting the orientation on Q. Let W be the corresponding Weyl group, generated by s_i for $i \in I$. Let Φ^+ be the set of postive roots and Φ^- be the set of negative roots. Fix an element $w \in W$. Arguably the most important case is when W is finite and w is the element of greatest length. We define the set

$$\Phi(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}.$$

The following important fact is standard

{prop:standard}

Proposition 8.1. *Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression of $w \in W$. For each $1 \le k \le l$, let $\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$. Then*

$$\Phi(w) = \{\beta_1, \beta_2, \ldots, \beta_l\}.$$

{def:convex}

Definition 8.2. [TW, Definition 1.8] *A convex preorder is a pre-order $\succ$ on Φ^+ such that,*

{cpo2}

(1) *For any equivalence class $\mathscr{C}$, any $a \in span_{\mathbb{R}_{\ge 0}} \mathscr{C}$ and any non-zero $x \in span_{\mathbb{Z}_{\ge 0}}\{\beta \in \Phi^+ \mid \beta \succ \mathscr{C}\}$, we have that $a + x \notin span_{\mathbb{Z}_{\ge 0}}\{\beta \in \Phi^+ \mid \beta \preceq \mathscr{C}\}$.*

{cpo3}

(2) *For any equivalence class $\mathscr{C}$, any $a \in span_{\mathbb{R}_{\ge 0}} \mathscr{C}$ and any non-zero $x \in span_{\mathbb{Z}_{\ge 0}}\{\beta \in \Phi^+ \mid \beta \prec \mathscr{C}\}$, we have that $a + x \notin span_{\mathbb{Z}_{\ge 0}}\{\beta \in \Phi^+ \mid \beta \succeq \mathscr{C}\}$.*

A convex order is a convex pre-order which is a total order on real roots.

{ex:convexorder}

Example 8.3. *Let $(V, \le)$ be a totally ordered $\mathbb{Q}$-vector space. Let $h : \mathbb{Q}\Phi \longrightarrow V$ be an injective linear transformation. For two positive roots α and β, say that $\alpha \prec \beta$ if $h(\alpha)/\operatorname{ht}(\alpha) < h(\beta)/\operatorname{ht}(\beta)$ and $\alpha \preceq \beta$ if $h(\alpha)/\operatorname{ht}(\alpha) \le h(\beta)/\operatorname{ht}(\beta)$. This defines a convex order on Φ.*

In the above example we can take $V = \mathbb{R}$ with the usual ordering to get the existence of many convex orders.

{lem:wconvex}

Lemma 8.4. *Fix a reduced expression $w = s_{i_1} \cdots s_{i_l}$ and let β_i be the root defined in Proposition 8.1. Then there exists a convex order $\prec$ such that $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_n$ and for any $\alpha \in \Phi^+ \setminus \Phi(w^{-1})$, $\alpha \succ \beta_l$.*

Proof. Choose a generic hyperplane $\mathfrak{h}$ in $\mathbb{R}\Phi$ such that the roots in $\Phi(w)$ and the roots in $\Phi^+ \setminus \Phi(w)$ are on opposite sides of $\mathfrak{h}$. Choose a linear map $h : \mathbb{R}\Phi \longrightarrow \mathbb{R}$, injective on $\mathbb{Q}\Phi$, such that $\mathfrak{h} = \ker h$ and $h(\Phi(w)) \subset \mathbb{R}_{<0}$. From h, the construction in Example 8.3 provides us with a convex order $\prec'$ on Φ^+. Now define our desired convex order $\prec$ by

- $\beta_1 \prec \beta_2 \prec \cdots \prec \beta_n$
- If $\alpha \in \Phi(w)$ and $\beta \notin \Phi(w)$ then $\alpha \prec \beta$
- If $\alpha, \beta \notin \Phi(w)$ then $\alpha \prec \beta$ if and only if $\alpha \prec' \beta$.

It is straightforward to check that this construction gives a convex order. □

Definition 8.5. *Let $\alpha \in \Phi^+$. An object (M, σ) of $\mathcal{C}_\alpha$ is $\prec$-cuspidal if whenever $\operatorname{Res}_{\lambda,\mu} M \ne 0$, we have that λ is a sum of roots less than or equal to α while μ is a sum of roots greater than or equal to α under $\prec$.*

Remark 8.6. Elsewhere, in [McNb, TW], this notion is called semicuspidal. Since we will only care about the case where α is a real root, the distinction between cuspidal and semicuspidal is irrelevant.

Theorem 8.7. [TW] *Let α be a real root and $\prec$ a convex order. Then there exists a unique self-dual simple $\prec$-cuspidal $R(\alpha)$-module, denoted $L(\alpha)$.*

The following theorem is [GLS11, Proposition 7.4] together with [McNb, Theorem 9.1]. But we will give a direct proof.

{cuspgm}

Theorem 8.8. *Let $s_{i_1}s_{i_2}\cdots s_{i_l}$ be a reduced expression for $w \in W$ and let $\prec$ be a convex order constructed from this reduced expression as in Lemma 8.4. For each k with $1 \le k \le l$, let $\beta_k = s_{i_1}\cdots s_{i_{k-1}}\alpha_{i_k}$ and let $L(\beta_k)$ be the corresponding cuspidal $R(\beta_k)$-module. Then*

$$[L(\beta_k)] = D(s_{i_1}\cdots s_{i_k}\omega_{i_k}, s_{i_1}\cdots s_{i_{k-1}}\omega_{i_k}),$$

the identity taking place in $\mathcal{A}_q(\mathfrak{n})$, identified with the Grothendieck group via Theorem 7.1.

Proof. Write D for $D(s_{i_1}\cdots s_{i_{k-1}}\omega_{i_k}, s_{i_1}\cdots s_{i_k}\omega_{i_k})$. By Theorem 4.2, $D \in \mathbf{B}^*$ so by [McNa, Theorem 12.7], D is the class of a self-dual simple module. Consider the classification of semicuspidal modules in terms of semicuspidal decompositions from [TW]. Since there is a unique irreducible cuspidal representation of $R(\beta_k)$, it suffices to show that

$$r_{\beta_l,\beta_k-\beta_l}(D) = 0$$

for all $l < k$.

Thus it suffices to show that $s_{i_1}\cdots s_{i_k}\omega_{i_k} - \beta_l$ is not a weight in $V(\omega_{i_k})$. Since β_l is a real root, it suffices to show

$$(s_{i_1}\cdots s_{i_k}\omega_{i_k}, \beta_l) \le 0.$$

By the Weyl group invariance of the pairing $(\cdot,\cdot)$ and the fact that $s_{i_l}(\alpha_{i_l}) = -\alpha_{i_l}$, this is equivalent to

$$(\omega_{i_k}, -s_{i_{k-1}}\cdots s_{i_{l+1}}\alpha_{i_l}) \le 0.$$

Since $s_{i_{k-1}}\cdots s_{i_l}$ is a reduced expression, $s_{i_{k-1}}\cdots s_{i_{l+1}}\alpha_{i_l}$ is a positive root, which thus has a nonnegative pairing with ω_{i_k}, completing the proof. $\square$

9. The quantum unipotent ring and its categorification

Fix $w \in W$. Choose a reduced expression for w. This determines an enumeration of $\Phi(w)$ by $\beta_1, \ldots, \beta_{\ell(w)}$ and some dual PBW vectors $E^*_{\beta_i}$. We will not give the usual definition of these in terms of the braid group action here, but will note that by Theorem 8.8 and [GLS13a, Proposition 7.4], $E^*_\beta = [L(\beta)]$.

Let $\mathcal{A}_q(\mathfrak{n}(w))$ be the $\mathbb{Z}[q,q^{-1}]$-subring of $\mathcal{A}_q(\mathfrak{n})$ generated by $E^*_{\beta_1}, \ldots, E^*_{\beta_l}$. It is known that this does not depend on the choice of a reduced decomposition.

Let $\pi = (\pi_1, \ldots, \pi_l)$ be a sequence of natural numbers. Associated to π is the element

$$E^*_\pi = (E^*_{\beta_1})^{\pi_1}\cdots(E^*_{\beta_l})^{\pi_l}.$$

Then this collection E^*_π is a $\mathbb{Z}[q,q^{-1}]$-basis of $\mathcal{A}_q(\mathfrak{n}(w))$.

Definition 9.1. *Let $\mathcal{C}_w(\nu)$ be the full subcategory of finite dimensional $R(\nu)$-modules such that whenever $\mathrm{Res}_{\lambda,\mu} M \neq 0$, $\lambda \in \mathbb{N}\Phi(w)$. Let $\mathcal{C}_w = \cup_\nu \mathcal{C}_w(\nu)$.*

Theorem 9.2. *The category $\mathcal{C}_\omega$ is the same as the category with the same name from* [KKKOb]*, and*

$$\bigoplus_{\nu\in\mathbb{N}I} K_0(\mathcal{C}_w(\nu)) \cong \mathcal{A}_q(\mathfrak{n}(w))$$

Proof. It is clear that the category $\mathcal{C}_w$ is closed under subquotients, extensions and induction product (to prove this last property, we need the Mackey filtration from [KL09]). It also contains the modules $L(\beta_k)$ since they are cuspidal. Therefore we have

$$\bigoplus_{\nu\in\mathbb{N}I} K_0(\mathcal{C}_w(\nu)) \supset \mathcal{A}_q(\mathfrak{n}(w))$$

To complete the proof, it will suffice to show that the dual PBW basis vectors E^*_π form a basis of $K_0(\mathcal{C}_w)$, since they form a basis of $\mathcal{A}_q(\mathfrak{n}(w))$. This will also solve the problem of matching notation with [KKKOb] since the categories considered here and there are both Serre subcategories.

The dual PBW basis vector E^*_π is the class of the proper standard module $\overline{\Delta}(\pi)$. By [McNb, Theorem 10.1(3)] (the argument works in all types), the change of basis matrix between the proper standard modules and the simple modules is unitriangular. Since the classes of the simple modules in $\mathcal{C}_w$ clearly form a basis of its Grothendieck group, the same is thus true for the classes of the proper standard modules, completing the proof. □

10. R-matrices

Since the polynomials $Q_{i,j}(u,v)$ appearing in the definition of the KLR algebras are all polynomials in $u-v$, the construction of R-matrices for KLR algebras in [KKK] applies.

Thus for every pair of modules X and Y, there is a nonzero morphism:

$$r_{X,Y} : X \longrightarrow Y.$$

Given any three modules X, Y, and Z, these R-matrices satisfy the Yang-Baxter equation

$$(r_{Y,Z}\circ \mathrm{id}_X)(\mathrm{id}_Y \circ r_{X,Z})(r_{X,Y}\circ \mathrm{id}_Z) = (\mathrm{id}_Z\circ r_{X,Y})(r_{X,Z}\circ \mathrm{id}_Y)(\mathrm{id}_X\circ r_{Y,Z}) \tag{10.1}$$ {ybe}

as well as the identity

$$(r_{Y\circ X,Z})(r_{X,Y}\circ \mathrm{id}_Z) = (\mathrm{id}_Z\circ r_{X,Y})(r_{X\circ Y,Z}). \tag{10.2}$$ {yb2}

The R-matrices also behave nicely with respect to the automorphism a, namely we have the identity

$$r_{a^*X,a^*Y} = a^* r_{X,Y}$$

Lemma 10.1. *Let X and Y be real modules such that $X\circ Y$ is simple. Then $X\circ Y \cong Y\circ X$ with a pair of inverse isomorphisms being given by $r_{X,Y}$ and $r_{Y,X}$.*

Proof. The isomorphism between $X\circ Y$ and $Y\circ X$ follows because by [KL09], the character map is injective at $q=1$. Thus once $X\otimes Y$ is known to be simple, $Y\otimes X$ must have only one simple Jordan-Holder factor by looking at its character at $q=1$. The R-matrices must provide isomorphisms by virtue of being nonzero maps between simple modules. By [KKK, Lemma 1.3.1(vi)], the R-matrices with spectral parameters satisfy $R_{Y,X}R_{X,Y}(v\otimes w) = (z-w)^t v\otimes w$ for vectors v and w of highest degree in X and Y respectively. Since $r_{Y,X}r_{X,Y} = (z-w)^{-s}R_{Y,X}R_{X,Y}|_{z=w=0}$ and is an isomorphism, it must be that $s=t$ and $r_{Y,X}r_{X,Y}$ is the identity (since it is a scalar multiple of the identity and preserves $v\otimes w$). Therefore the R-matrices $r_{X,Y}$ and $r_{Y,X}$ are inverses of each other. □

For such X and Y, there are integers $\Lambda(X,Y)$ and $\Lambda(Y,X)$ such that the modules $q^{\Lambda(X,Y)}X\circ Y$ and $q^{\Lambda(Y,X)}Y\circ X$ are self-dual simple modules (this is denoted $\tilde{\Lambda}$ in [KKKOb]). Now consider the diagram

$$q^{\Lambda(X,Y)}X\circ Y \underset{r_{Y,X}}{\overset{r_{X,Y}}{\rightleftarrows}} q^{\Lambda(Y,X)}Y\circ X.$$

We define $X\odot Y$ to be the direct limit of this diagram, it is thus a self-dual irreducible module which is canonically isomporphic to both $X\circ Y$ and $Y\circ X$ up to grading shift.

Now given $(X,\sigma)\in\mathcal{C}_\lambda$ and $(Y,\tau)\in\mathcal{C}_\mu$ with X and Y as above, the following diagram commutes

$$\begin{tikzcd}
q^{\Lambda(X,Y)}X\circ Y \arrow[r, shift left, "r_{X,Y}"] & q^{\Lambda(Y,X)}Y\circ X \arrow[l, shift left, "r_{Y,X}"] \\
q^{\Lambda(X,Y)}a^*(X\circ Y) \arrow[u, "\sigma\circ\tau"] \arrow[r, shift left, "a^*r_{X,Y}"] & q^{\Lambda(Y,X)}a^*(Y\circ X) \arrow[u, "\tau\circ\sigma"'] \arrow[l, shift left, "a^*r_{Y,X}"]
\end{tikzcd}$$

so there is a canonical element $(X\odot Y,\sigma\odot\tau)$ in $\mathcal{C}_{\lambda+\mu}$ which does not depend on the order of the factors up to canonical isomorphism.

Now suppose we have a family of self-dual simple modules $X_1,\ldots,X_n$ such that $X_i\circ X_j$ is simple for all i,j. Then we define $\bigodot_{i=1}^n X_i$ inductively by

$$\bigodot_{i=1}^n X_i = \left(\bigodot_{i=1}^{n-1} X_i\right)\odot X_n.$$

Via the R-matrices, using (10.1) and (10.2), this module is canonically isomorphic to a grading shift of $X_{\sigma(1)}\circ\cdots\circ X_{\sigma(n)}$ for any permutation $\sigma\in S_n$.

If in addition there are isomorphisms $\sigma_i : a^*X_i \longrightarrow X_i$ satisfying (5.2), then by iterating the case of two modules, we obtain an object

$$\left(\bigodot_{i=1}^n X_i, \bigodot_{i=1}^n \sigma_i\right)$$

which does not depend on the order of the factors up to canonical isomorphism.

11. Reduction modulo p

We shall need to make some arguments involving reduction modulo p. For this we need a p-modular system. Since we can work over any field with n n-th roots of unity, it is easy to find such a system. For example, we can take $F=\mathbb{Q}(\zeta_n)$, $\mathcal{O}=\mathbb{Z}[\zeta_n]$ and let π be a place of F over p (recall that we assume p does not divide n). Then $(F_\pi,\mathcal{O}_\pi,\mathbb{F}_p[\zeta_n])$ is a p-modular system. The KLR algebras are known to be free over $\mathbb{Z}$ [Mak15], so the standard theory of reduction modulo p can be applied.

The following diagram of isomorphisms commutes, where d is the decomposition map and the subscripts on the $\mathbf{k}^*$s and γ^*s refer to the characteristic of the field k used to define them.

$$\begin{tikzcd} \mathbf{k}_0^* \arrow[rr, "d"] \arrow[dr, "\gamma_0^*"'] & & \mathbf{k}_p^* \arrow[dl, "\gamma_p^*"] \\ & \mathbf{f}^* & \end{tikzcd}$$

A module M is said to be *real* if $M \circ M$ is irreducible. {mexists}

Proposition 11.1. *Let $\lambda \in P^+$ and $\mu \leq \eta$ two elements of $W\lambda$. Then there exists a real self-dual simple $R(\eta - \mu)$-module $M(\eta, \mu)$ satisfying*

$$\gamma^*([M(\mu, \eta)]) = D(\mu, \eta).$$

Furthermore, the reduction of $M(\mu, \eta)$ modulo any prime remains irreducible.

Proof. By Theorem 4.2, $D(\mu, \eta) \in \mathbf{B}^*$. Therefore a self-dual simple $R(\eta - \mu)$-module $M(\eta, \mu)$ must exist satisfying $\gamma^*([M(\mu, \eta)]) = D(\mu, \eta)$. This module is real because by Lemma 4.4, $D(\mu, \eta)^2 \in q^{\mathbb{Z}}\mathbf{B}^*$. It remains irreducible when reduced modulo p because the decomposition map commutes with the isomorphism to $\mathcal{A}_q(\mathfrak{n})$. □

There are fundamentally three places in [KKKOb] where the assumption that the ground field k has characteristic zero is used. This is in their proof that all generalised minors lie in $\mathbf{B}^*$, as well as Theorems 4.28 and 4.30 of [KKKOb]. In the rest of this section, we show how to prove these results in arbitrary charcteristic, ensuring that all results in [KKKOb] are valid in arbitrary characteristic.

We know that all generalised minors $D(\mu, \zeta)$ lie in $\mathbf{B}^*$ by Theorem 4.2 and [McNa]. The other two results which we need to prove appear as Theorem 11.2 and Corollary 11.5 below.

Given any two modules M and N, we define $M \diamond N$ to be the head of $M \circ N$.

The following result is [KKKOb, Theorem 4.28] in the case when k is of characteristic zero. {4.28}

Theorem 11.2. *Let $\lambda \in P^+$ and $\mu_1, \mu_2, \mu_3 \in W\lambda$ such that $\mu_1 \preceq \mu_2 \preceq \mu_3$. Then*

$$M(\mu_1, \mu_2) \diamond M(\mu_2, \mu_3) \cong M(\mu_1, \mu_3).$$

Remark 11.3. It is possible to prove this in all characteristics from the characteristic zero result by an argument using reduction modulo p. However we give a uniform proof.

Proof. By Lemma 4.3, we get

$$\mathrm{Res}_{\mu_2 - \mu_1, \mu_3 - \mu_2} M(\mu_1, \mu_3) \cong M(\mu_1, \mu_2) \otimes M(\mu_2, \mu_3).$$

By adjunction there is thus a nonzero morphism from $M(\mu_1, \mu_2) \circ M(\mu_2, \mu_3)$ to $M(\mu_1, \mu_3)$. Since $M(\mu_1, \mu_2)$ is real, [KKKO15, Theorem 3.2] implies that $M(\mu_1, \mu_2) \circ M(\mu_2, \mu_3)$ has an irreducible head, which must thus be $M(\mu_1, \mu_3)$. □ {reducemodp}

Lemma 11.4. *Let M and N be modules for a characteristic zero KLR algebra and let M_p and N_p denote reductions of M and N modulo the prime p. Suppose that $M_p \circ M_p$ and N_p are simple. Then*

$$\deg r_{M,N} = \deg r_{M_p, N_p}.$$

Proof. Since $M_p \circ M_p$ is simple, the same is true of $M \circ M$. It is immediate from [KKKO15, Theorem 3.2] that the spaces $\mathrm{Hom}(M \circ N, N \circ M)$ and $\mathrm{Hom}(M_p \circ N_p, N_p \circ M_p)$ are one dimensional, spanned by $r_{M,N}$ and r_{M_p, N_p} respectively. The morphism $r_{M,N}$ can be reduced modulo p to give a nonzero morphism in $\mathrm{Hom}(M_p \circ N_p, N_p \circ M_p)$ of the same degree. By the one-dimensionality of this homomorphism space, we have our desired result. □

As a Corollary, we are able to prove [KKKOb, Proposition 4.30] in all characteristics:

{4.30}

Corollary 11.5. *Let $x \in W$ and $i \in I$ be such that $xs_i > x$ in Bruhat order and $x\omega_i \neq \omega_i$. Let $X = M(xs\omega_i, x\omega_i)$ and $Y = M(x\omega_i, \omega_i)$. Then*

$$\deg r_{X,Y} + \deg r_{Y,X} = 2.$$

Proof. By Proposition 11.1 and Lemma 11.4, we reduce ourselves to the case where the ground field k is of characteristic zero, which is [KKKOb, Proposition 4.30]. □

As a consequence of these results, all results proved in [KKKOb] are valid over an arbitrary ground field

12. Quantum Cluster Algebras

{qcas}

Here we give the definition of a skew-symmetrisable quantum cluster algebra. Some of our powers of q are different from those which appear elsewhere in the literature. We make this choice so that we do not have to ever extend scalars to $\mathbb{Q}[q^{\pm 1/2}]$. The specialisation at $q = 1$ recovers the classical commutative cluster algebra.

The quivers that appear from now on will always refer to combinatorial data used in cluster mutation. They bear no relation to the quiver used to define the KLR algebras, which can be safely forgotten about.

Let $\mathbf{ex} \subset S$ be two finite sets. A cluster matrix is an integer matrix B with rows labelled by S and columns labelled by $\mathbf{ex}$. We call elements of S vertices and elements of $\mathbf{ex}$ exchangable vertices.

Let $\Lambda = (\lambda_{ij})_{i,j\in S}$ be a skew-symmetric integer matrix. We say that the pair (Λ, B) is *compatible* if

$$\Lambda B = -2E$$

where $E = (e_{st})$ is a matrix with $e_{st} \geq 0$ for all s, t and $e_{st} > 0$ if and only if $s = t \in \mathbf{ex}$. We write e_s for the integer e_{ss}. If (Λ, B) is compatible, then the principal part of B is automatically skew-symmetrisable.

The mutation of (Λ, B) in the direction $k \in \mathbf{ex}$ is the pair (Λ', B') where

$$\lambda'_{st} = \begin{cases} \lambda_{st} & \text{if } s, t \neq k \\ \sum_i \max(b_{ik}, 0)\lambda_{it} & \text{if } s = k \neq t, \end{cases}$$

and

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{ij}|}{2} & \text{otherwise.} \end{cases}$$

It is easily checked that the mutation of a compatible pair is compatible.

The based quantum torus associated with Λ is the $\mathbb{Z}[q^{\pm 1}]$-algebra $\mathcal{T}(\Lambda)$ generated by $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$ subject to the relations

$$X_i X_j = q^{\lambda_{ij}} X_j X_i.$$

$\mathcal{T}(\Lambda)$ is an Ore domain and embeds into its field of skew-fractions..

Let $D = (d_{ij})$ be a symmetric matrix such that $\Lambda \equiv D \pmod 2$. Then $\mathcal{T}(\Lambda)$ has a bar involution where $\overline{q} = q^{-1}$, $\overline{X_i} = X_i$, $\overline{X_i^{-1}} = X_i^{-1}$ and

$$\overline{AB} = q^{{}^t\mathbf{a}D\mathbf{b}}\overline{B} \cdot \overline{A}$$

where A and B are monomials of degree $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ respectively.

Let $\Lambda = (\lambda_{ij})_{i,j \in S}$ be a skew-symmetric integer matrix and $\{d_i\}_{i \in S}$ a family of elements in Q^+ such that $\lambda_{ij} \cong (d_i, d_j) \pmod 2$ for all $i, j \in S$. Let $Y_i \in \mathcal{A}_q(\mathfrak{n}(w))_{d_i}$ for $i \in S$ be a family of elements such that

$$Y_i Y_j = q^{\lambda_{ij}} Y_j Y_i.$$

We call such a family of elements Λ-commuting.

Given such a choice of Λ-commuting elements, make a choice of identification $S \cong \{1, 2, \dots, m\}$. Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{Z}^m$. Define

$$Y^{\mathbf{a}} = (q^{1/4})^{(\sum_i a_i d_i, \sum_i a_i d_i) - \sum_i a_i (d_i, d_i) + 2\sum_{i>j} a_i a_j \lambda_{ij}} Y_1^{a_1} Y_2^{a_2} \cdots Y_m^{a_m}.$$

This is an element of the noncommutative field of fractions of $\mathcal{A}_q(\mathfrak{n}(w))$ that does not depend on the ordering of the indexing set S. If $\mathbf{a} \in \mathbb{N}^m$ then this element lies in $\mathcal{A}_q(\mathfrak{n}(w))$. The condition $\lambda_{ij} \cong (d_i, d_j) \pmod 2$ implies that the exponent of q is an integer. If each Y_i is self-dual, then so is $Y^{\mathbf{a}}$.

Let (Λ, B) be a compatible pair and $\mathcal{A}$ a graded $\mathbb{Z}[q^{\pm 1}]$-algebra contained in a skew field $\mathcal{K}$. We say that $(\{Y_s\}_{s \in S}, \Lambda, B)$ is a quantum seed in $\mathcal{A}$ if $\{Y_s\}$ is a Λ-commuting family of elements of $\mathcal{A}$ such that the induced algebra homomorphism $\mathcal{T}(\Lambda) \to \mathcal{A}$ is injective. The set $\{Y_s\}_{s \in S}$ is also called the cluster, and the Y_s are the cluster variables. If $s \in \mathbf{ex}$ then the corresponding cluster variable is called exchangable, otherwise frozen. The elements $Y^{\mathbf{a}}$ are the quantum cluster monomials.

Fix $s \in \mathbf{ex}$. Define $a^+(s)_t = \max(b_{ts}, 0)$ and $a^-(s)_t = \max(-b_{ts}, 0)$. This defines two sequences $\mathbf{a}^+(s)$ and $\mathbf{a}^+(s)$ of integers indexed by S. The mutation variable Y_s' is then defined by

$$Y_s Y_s' = Y^{\mathbf{a}^+(s)} + q^{e_s} Y^{\mathbf{a}^-(s)}. \tag{12.1}$$ {12.n}

Then $(\{Y_t\}_{t \in S} \cup \{Y_s'\} \setminus \{Y_s\}, \Lambda', B')$ is also a quantum seed in $\mathcal{K}$.

Definition 12.1. *Let $\mathcal{I}$ be a quantum seed in $\mathcal{A}$. The quantum cluster algebra $\mathcal{A}(\mathcal{I})$ associated to $\mathcal{I}$ is the $\mathbb{Z}[q^{\pm 1}]$-subalgebra of $\mathcal{K}$ generated by all quantum cluster variables in all quantum seeds obtained from $\mathcal{I}$ by all sequences of mutations.*

13. The initial quiver

{subsec}

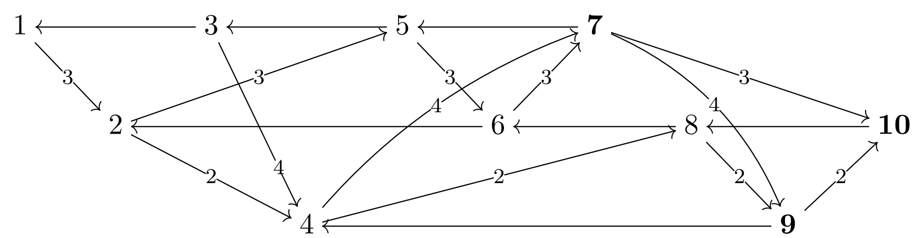
We now give a construction of a quantum cluster algebra associated to an element $w \in W$. The construction will a priori depend on a choice of reduced expression for w, but ultimately we will show that this quantum cluster algebra does not depend on this choice.

Let $w = s_{i_1} \dots s_{i_m}$ be a reduced expression for $w \in W$. To this data, we now define a quiver $Q(i_1, \dots, i_m)$. Consider an array indexed by $\{1, \dots, m\} \times I$. We place a vertex at each point in the array of the form (t, i_t). We place a horizonal arrow from the vertex (a, i) to the vertex (b, i) if $a > b$ and there is no vertex (c, i) with $a > c > b$. The other arrows between rows form a zigzag pattern: We place $-i \cdot j$ arrows from (a, i) to (b, j) if $a < b$ and there is no (c, j) with the properties that (i) $c > b$ and (ii) there is no vertex (d, i) with $c > d > b$. For each $i \in I$, the vertex of the form (a, i) with largest a is decreed to be frozen.

For example suppose that our Cartan matrix is

$$\begin{pmatrix} 2 & -3 & -4 \\ -3 & 2 & -2 \\ -4 & -2 & 2 \end{pmatrix}$$

and that our reduced word is $s_i s_j s_i s_k s_i s_j s_i s_j s_k s_j$, where i, j and k are used to label the rows of the Cartan matrix, in that order. Then the quiver $Q(i, j, i, k, i, j, i, j, k, j)$ is



where a number on an arrow means that there are that many arrows between the vertices. Frozen vertices are depicted in bold.

To the vertex (t, i_t), we associate the cluster variable

$$Y_t := D(s_{i_1} \cdots s_{i_t} \omega_{i_t}, \omega_{i_t}).$$

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