Three regimes for torsion in Schubert and quiver varieties

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The study of torsion in the stalks and costalks of interesection cohomology sheaves of Schubert varieties and (Lusztig) quiver varieties is a problem that is of fundamental importance in modular representation theory. We identify and discuss three different qualitative types of behaviour which can occur, and discuss examples of each such type of behaviour. These possibilities are for there to be no torsion, tamely controlled torsion and exponential growth of torsion.

Let $\mathfrak{g}$ be a symmetrisable Kac-Moody Lie algebra and w an element of its Weyl group. We let $A_q(\mathfrak{n}(w))$ be the quantised coordinate ring which is a quantisation of the coordinate ring of the unipotent group whose Lie algebra is the span of the positive roots α such that $w\alpha$ is negative. For each prime p (including $p = 0$) this algebra has a basis, called the dual p-canonical basis, obtained via categorification. This categorification either uses indecomposable summands of Lusztig sheaves with coefficients in a field of characteristic p, or representations of KLR algebras (with a diagram automorphism if necessary) over a field of characteristic p.

Building on recent work of Kang, Kashiwara, Kim and Oh [KKKO1, KKKO2], who prove the $p = 0$ version of the below theorem when $\mathfrak{g}$ is symmetric, we are able to prove the following:

Theorem 1. [Mc] *The quantised coordinate ring $A_q(\mathfrak{n}(w))$ has the structure of a quantum cluster algebra in which the cluster monomials lie in the dual p-canonical basis.*

The quantum cluster algebra structure on $A_q(\mathfrak{n}(w))$ is defined independently of p, in terms of an initial seed consisting of quantum generalised minors. This theorem thus provides a region where the dual p-canonical bases for different p all agree, or equivalently that certain decomposition numbers for KLR algebras are trivial. By [W1], this is connected to the nonexistence of torsion in the intersection cohomology of Lusztig quiver varieties.

We now turn our example to examples of torsion. For primes p dividing the off-diagonal entries of the Cartan matrix, the existence of p-torsion in Schubert varieties has been known since near the beginning of the theory of intersection cohomology. More interesting are the examples of 2-torsion in the A_7 and D_4 flag varieties discovered by Braden [B], and p-torsion in the A_{4p-1} flag variety discovered by Polo (unpublished).

We provide some examples of tame families of torsion, which are smoothly equivalent to well-understood examples of torsion in quiver varieties of affine type. These families provide examples of p-torsion in quiver varieties of type A_5 and D_4 (the former generalising the famous singularity of Kashiwara-Saito) and p-torsion within a single left cell in the flag variety of type A_{6p-1} (generalising an example of Williamson [W3]).

Now we turn to examples of examples of torsion which grow exponentially, the first families of which were constructed by Williamson [W2] using the diagrammatic calculus of Soergel bimodules. In joint work with Kontorovich and

Williamson, we are able to use recent advances in analytic number theory (the affine sieve, or the recent progress on Zaremba's conjecture) to prove the theorem below. We say there exists p-torsion in a variety X if there exists p-torsion in a stalk or costalk of the intersection cohomology complex $IC(X;\mathbb{Z})$.

Theorem 2. [KMW] *There exists a constant $c > 1$ such that the set*

$$\{p > c^N \mid p \text{ is prime and there exists } p\text{-torsion in a Schubert variety in } SL_N\}$$

is nonempty for all sufficiently large N.

There is a more precise version with a lower bound on the growth of the size of this set in [KMW].

References

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