NON-PERVERSE PARITY SHEAVES ON THE FLAG VARIETY

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Abstract. We give examples of non-perverse parity sheaves in Schubert varieties for all primes.

1. Introduction

The notion of a parity sheaf was introduced in [JMW] and has become an important object in modular geometric representation theory. An even/odd sheaf on a complex variety X with coefficients in a field k is an object of $D^b(X;k)$ whose star and shriek restrictions to all points only have even/odd cohomology. A parity sheaf is a direct sum of an even and an odd sheaf. (We only consider parity sheaves for the zero pariversity in this paper).

In this paper, we take X to be the variety of all complete flags in $\mathbb{C}^N$, and only consider sheaves which are constructible with respect to the stratification by Schubert varieties. Then by [JMW, Theorem 4.6], for each $w \in S_n$, there exists a unique indecomposable parity sheaf $\mathcal{P}_w$ whose support is the Schubert variety X_w. Up to homological shift, these constitute all Borel-constructible parity sheaves on the flag variety X.

Let p be the characteristic of k. We provide the first examples of parity sheaves on Schubert varieties which are not perverse for primes $p > 2$. Examples for $p = 2$ were recently constructed in [LW]. Our family of examples also includes parity sheaves which are arbitrarily non-perverse. We are not able to provide examples with p greater than the Coxeter number, but we expect that such examples exist. Such examples are of interest thanks to [AR].

Our examples generalise constructions of Kashiwara and Saito [KS], Polo (unpublished), and the author and Williamson [MW].

2. Statement of the result

Let p be a prime. Let d and l be positive integers such that $p^d \geq l \geq 3$. Let $q = p^d$. Define the following permutation $y \in S_{q(l+2)}$:

Date: November 8, 2018.

$$y(j)=\begin{cases}(l+1)q & \text{if } j=1\\ q+1-j & \text{if } 2\le j\le q\\ (l+2)q & \text{if } j=q+1\\ (l+2)q+1-j & \text{if } q+2\le j<(l+1)q \text{ and } j\not\equiv 0,1 \pmod q\\ (l+2)q-j & \text{if } q+2\le j<(l+1)q \text{ and } j\equiv 0 \pmod q\\ (l+2)q+2-j & \text{if } q+2\le j<(l+1)q \text{ and } j\equiv 1 \pmod q\\ 1 & \text{if } j=(l+1)q\\ (2l+3)q-j & \text{if } (l+1)q<j<(l+2)q\\ q+1 & \text{if } j=(l+2)q.\end{cases}$$

Let $\mathcal{P}_y$ be the indecomposable parity sheaf supported on the Schubert variety X_y with coefficients in $\mathbb{F}_p$, extending the constant sheaf shifted by $\dim(X_y)$. Our theorem is:

Theorem 2.1.

$$\mathcal{P}_y \ncong {}^p\tau_{\le l-3}(\mathcal{P}_y).$$

3. Intersection Forms

Our main tool is the following result which computes the multiplicities of a direct summand via the rank of a bilinear form. First we make a definition.

If A is an indecomposable object in a Krull-Schmidt category and X is any object, write $m(A,X)$ for the number of times A appears as a direct summand of X.

Proposition 3.1. [JMW, Proposition 3.2] *Let k be a local ring. Let $\pi:\tilde{Y}\longrightarrow Y$ be a proper resolution of singularities. Let $y\in Y$ be a point, and suppose that the fibre $F=\pi^{-1}(y)$ is smooth. Write i for the inclusion of y in Y. Let n be the dimension of $\widetilde{Y}$, d the dimension of F and m be an integer. Let B be the pairing*

$$H^{2d-n-m}(F)\times H^{2d-n+m}(F)\to H^{2d}(F)$$

given by $B(\alpha,\beta)=\alpha\cup\beta\cup e$, where e is the Euler class of the normal bundle to F in Y. Then

$$m(i_*\underline{k}[m],\pi_*\underline{k}[n])=\operatorname{rank}(B).$$

Proof. By general results about multiplicities of indecomposable objects in Krull-Schmidt categories, the multiplicity $m(i_*\underline{k}[m],\pi_*\underline{k}[n])$ is equal to the rank of the pairing

$$\operatorname{Hom}(i_*\underline{k}[m],\pi_*\underline{k}_{\widetilde{Y}}[n])\times\operatorname{Hom}(\pi_*\underline{k}_{\widetilde{Y}}[n],i_*\underline{k}[m])\to\operatorname{End}(i_*\underline{k}[m])\cong k. \tag{3.1}$$

The following commutative diagram appears in the proof of [JMW, Lemma 3.4] and arises through applying base change and adjunctions. The pairing B' arises through composition with the canonical map from $\omega_F[-n]$ to $\underline{k}_F[n]$ (which comes from applying the canonical natural transformation $i^!\to i^*$ to $\underline{k}_{\widetilde{Y}}$).

$$\begin{array}{ccc} \operatorname{Hom}(\underline{k}_F[m], \omega_F[-n]) \times \operatorname{Hom}(\underline{k}_F[n], \omega_F[m]) & \xrightarrow{B'} & \operatorname{Hom}(\underline{k}_F[m], \omega_F[m]) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(i_*\underline{k}[m], \pi_*\underline{k}_{\widetilde{Y}}[n]) \times \operatorname{Hom}(\pi_*\underline{k}_{\widetilde{Y}}[n], i_*\underline{k}[m]) & \longrightarrow & \operatorname{End}(i_*\underline{k}[m]) \end{array}$$

Since F is smooth, $\omega_F \cong \underline{k}_F[2d]$ and the canonical morphism from $\omega_F[-n]$ to $\underline{k}_F[n]$ is identified with the Euler class e. Therefore the pairing (3.1) is identified with the one stated in the proposition, completing the proof. □

4. Geometry

Let x be the permutation

$$\begin{pmatrix} J & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & J & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & J & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & J \end{pmatrix}$$

where J is the $q \times q$ antidiagonal matrix.

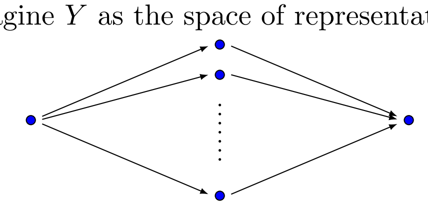
We compute the slice to the Schubert variety X_y at the point x. Using the techniques discussed in [W], this is given by matrices of the form

$$\begin{pmatrix} 0 & & & & & \\ A_l & & & & & \\ \vdots & & & & & \\ A_2 & & & & & \\ A_1 & & & & & \\ 0 & B_1 & B_2 & \cdots & B_l & 0 \end{pmatrix}$$

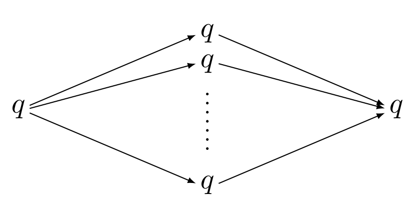
subject to the conditions

$$B_i J A_i = 0.$$

Let Y be this slice. Reimagine Y as the space of representations of the quiver



of dimension vector



Let

$$\widetilde{Y} = \{(h, \ell_1, \dots, \ell_l, \ell, A_1, \dots, A_l, B_1', \dots, B_l') \mid h \in Gr(q-1, q),$$
$$\ell_1, \dots, \ell_l, \ell \in Gr(1, q), A_i \in \mathrm{Hom}(\mathbb{C}^q/h, \ell_i), B_i \in \mathrm{Hom}(\mathbb{C}^q/\ell_i, \ell)\}$$

Then $\widetilde{Y}$ is the total space of a vector bundle $\mathcal{E}$ on $Z := (\mathbb{P}^{q-1})^{k+2}$. Motivated by Proposition 3.1, we consider the pairing

$$\langle \cdot, \cdot \rangle : H^{2(q-k)}(Z) \times H^{2(q-2)}(Z) \longrightarrow H^{2(k+2)(q-1)}(Z)$$

given by

$$\langle \sigma, \tau \rangle = \sigma \cup \tau \cup e(\mathcal{E}).$$

We have

$$H^*(Z) \cong \mathbb{Z}[w]/(w^q) \otimes (\bigotimes_{i=1}^{l} \mathbb{Z}[a_i]/(a_i^q)) \otimes \mathbb{Z}[z]/(z^q)$$

where the variables w and z come from the factors of Z corresponding to the choices of h and ℓ in $\widetilde{Y}$ respectively, while the a_i come from the choice of ℓ_i.

Lemma 4.1. *The Euler class of $\mathcal{E}$ is given by*

$$e(\mathcal{E}) = \prod_{i=1}^{l} \left((a_i + w) \sum_{j=0}^{q-1} a_i^j z^{q-1-j} \right). \tag{4.1}$$

Proof. The bundle $\mathcal{E}$ is naturally a direct sum of $2l$ vector bundles

$$\mathcal{E} \cong \bigoplus_{i=1}^{l} \mathcal{A}_i \oplus \bigoplus_{i=1}^{l} \mathcal{B}_i,$$

where the bundles $\mathcal{A}_i$ and $\mathcal{B}_i$ correspond to the choice of A_i and B_i in $\widetilde{Y}$. The bundle $\mathcal{A}_i$ is a line bundle with first Chern class $a_i + w$.

On $\mathbb{P}^{q-1}$, write $\mathcal{L}$ for the tautological line bundle, and $\mathcal{T}$ for the trivial vector bundle of rank q. Then, restricted to the appropriate $\mathbb{P}^{q-1} \times \mathbb{P}^{q-1}$, we have

$$\mathcal{B}_i \cong p_1^*(\mathcal{T}/\mathcal{L})^* \otimes p_2^* \mathcal{L}.$$

The total Chern class of $(\mathcal{T}/\mathcal{L})^*$ is given by

$$c((\mathcal{T}/\mathcal{L})^*) = \sum_{j=1}^{\infty} a_i^j.$$

The computation of $e(\mathcal{B}_i)$ proceeds via (4.2) below, whose proof follows easily using the splitting principle:

Let V be a rank n vector bundle and L be a line bundle. Then

$$e(V \otimes L) = \sum_{i=0}^{n} c_i(V) c_1(L)^{n-i}. \tag{4.2}$$

Using this, we obtain

$$e(\mathcal{B}_i) = \sum_{j=0}^{q-1} a_i^j z^{q-1-j}.$$

Together with the formula for the Euler class (=first Chern class) of $\mathcal{A}_i$, this implies the formula in the lemma. □

When computing the intersection pairing $\langle \cdot, \cdot \rangle$ between two monomials in $H^*(Z)$, one notices that it only depends on the power of w in their product. If this power is j, then the pairing takes the value $\binom{l}{q-1-j}$, for this is the number of ways of choosing terms in the product formula (4.1) for $e(\mathcal{E})$ which produce the right power of w, and once these choices are made, the rest are uniquely determined.

Deleting irrelevant rows, the matrix for our intersection form becomes

$$M = \begin{pmatrix} l & \binom{l}{2} & \binom{l}{3} & \cdots & & & l & 1 & & & & \\ 1 & l & \binom{l}{2} & \binom{l}{3} & \cdots & & & l & 1 & & & \\ & 1 & l & \binom{l}{2} & & & & & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & & & & & l & 1 & \\ & & & 1 & l & \binom{l}{2} & \cdots & & & & l & 1 \\ & & & & 1 & l & \binom{l}{2} & \cdots & & & & l \end{pmatrix} \tag{4.3}$$

This matrix has $q-1$ columns and $q-l+1$ rows.

5. FIN

Lemma 5.1. *The matrix M from (4.3) has rank $q-l+1$ over $\mathbb{Q}$ and $q-l$ over $\mathbb{F}_p$.*

Proof. Identify the rows of M with the sequence of polynomials

$$\begin{aligned} &(1+x)^l - 1 \\ &x(1+x)^l \\ &x^2(1+x)^l \\ &\vdots \\ &x^{q-l-1}(1+x)^l \\ &x^{q-l}(1+x)^l - x^q. \end{aligned}$$

If there is a linear dependence, then $A + Bx^q$ is divisible by $(1+x)^l$ for some constants A and B. Over $\mathbb{Q}$, this is impossible since $A + Bx^q$ has distinct roots over $\mathbb{C}$. This computes the rank over $\mathbb{Q}$.

Over $\mathbb{F}_p$, the polynomial $(1+x)^l$ divides $x^q + 1$, which easily leads to a linear dependence amongst the rows of M. It is obvious that the rank is at least $q-l-2$, completing the proof in this case. □

We now come to the proof of Theorem 2.1.

Proof. Let $\pi : \widetilde{Y} \longrightarrow \widetilde{Y}$ be the canonial morphism between the spaces $\widetilde{Y}$ and Y defined in §4 and let n be their common dimension. Decompose $\pi_*\underline{\mathbb{Z}}_p[n]$ into indecomposables

$$\pi_*\underline{\mathbb{Z}}_p[n] \cong \bigoplus_i \mathcal{P}_i^{n_i}.$$

Each $\mathcal{P}_i \otimes \mathbb{F}_p$ will be an indecomposable parity sheaf.

By Proposition 3.1 and Lemma 5.1, the multiplicity of $i_*\underline{k}[l-2]$ in $\pi_*\underline{k}[n]$ is $q-l+1$ when $k = \mathbb{Q}_p$ and $q-l$ when $k = \mathbb{F}_p$. Therefore there exists a unique i such that

$$m(i_*\underline{\mathbb{Q}}_p[l-2], \mathcal{P}_i \otimes \mathbb{Q}_p) = 1.$$

This setup is symmetric under the action of the symmetric group S_l. So by uniqueness of i, $\mathcal{P}_i$ is S_l-invariant.

There are only two S_l-invariant intermediate strata in Y. One is where $A_1 = \cdots = A_l = 0$ and the other is $B_1 = \cdots = B_l = 0$. They can be treated similarly.

In each case the stratum has an even resolution by the total space of a vector bundle $\mathcal{F}$ over $\mathbb{P}^{q-1}$, where the zero section is the fibre over $0 \in Y$, and $e(\mathcal{F})$ is a power of $c_1(\mathcal{O}(1))$.

Thus in computing the intersection form, once irrelevant rows are deleted, one is left with the antidiagonal matrix J, which has the same rank over $\mathbb{Q}$ and $\mathbb{F}_p$.

By a similar argument to the decomposition of $\pi_*\underline{\mathbb{Z}}_p[n]$, the parity sheaves supported on these strata cannot be the $\mathcal{P}_i$ above.

Therefore $\mathcal{P}_i$ is the parity extension of the shifted constant sheaf on Y. As the stalk of $\mathcal{P}_i$ at 0 is free over $\mathbb{Z}_p$ and satisfies a parity vanishing peroperty, it is $\mathbb{Z}_p$ in degree $l-2$. Therefore $\mathcal{P}_i \otimes \mathbb{F}_p$ has nonzero stalk cohomology at 0 in degree $l-2$. Thus

$$\mathcal{P}_i \otimes \mathbb{F}_p \ncong {}^p\tau_{\leq l-3}(\mathcal{P}_i \otimes \mathbb{F}_p).$$

Since $\mathcal{P}_i \otimes \mathbb{F}_p$ is the restriction of $\mathcal{P}_y$ to the slice Y, the analogous statement holds for $\mathcal{P}_y$, completing the proof. □

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