NON-PERVERSE PARITY SHEAVES ON THE FLAG VARIETY

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Abstract. We give examples of non-perverse parity sheaves on Schubert varieties for all primes.

1. Introduction

The notion of a parity sheaf was introduced in [JMW] and has since become an important object in modular geometric representation theory. An even/odd sheaf on a complex variety $X$ with coefficients in a field $k$ is an object of $D^b(X; k)$ whose star and shriek restrictions to all points only have even/odd cohomology. A parity sheaf is a direct sum of an even and an odd sheaf. (We only consider parity sheaves for the zero pariversity in this paper).

In this paper, we take $X$ to be the variety of all complete flags in $\mathbb{C}^N$, and only consider sheaves which are constructible with respect to the stratification by Schubert varieties. Then by [JMW, Theorem 4.6], for each $w \in S_n$, there exists a unique indecomposable parity sheaf $E_w$ whose support is the Schubert variety $X_w$. Up to homological shift, these constitute all Borel-constructible parity sheaves on the flag variety $X$.

Let $p$ be the characteristic of $k$. We provide the first examples of parity sheaves on Schubert varieties which are not perverse for primes $p > 2$. Examples for $p = 2$ were recently constructed in [LW]. Our family of examples also includes parity sheaves which are arbitrarily non-perverse. We are not able to provide examples with $p$ greater than the Coxeter number, but we expect that such examples exist. Such examples are of interest thanks to [AR].

Our examples generalise constructions of Kashiwara and Saito [KS], Polo (unpublished), and the author and Williamson [MW].

2. Statement of the result

Let $p$ be a prime. Let $d$ and $l$ be positive integers such that $p^d \geq l \geq 3$. Let $q = p^d$. Define the following permutation $y \in S_{q(l+2)}$: 

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\[ y(j) = \begin{cases} 
(l + 1)q & \text{if } j = 1 \\
q + 1 - j & \text{if } 2 \leq j \leq q \\
(l + 2)q & \text{if } j = q + 1 \\
(l + 2)q + 1 - j & \text{if } q + 2 \leq j < (l + 1)q \text{ and } j \not\equiv 0, 1 \pmod{q} \\
(l + 2)q - j & \text{if } q + 2 \leq j < (l + 1)q \text{ and } j \equiv 0 \pmod{q} \\
(l + 2)q + 2 - j & \text{if } q + 2 \leq j < (l + 1)q \text{ and } j \equiv 1 \pmod{q} \\
1 & \text{if } j = (l + 1)q \\
(2l + 3)q - j & \text{if } (l + 1)q < j < (l + 2)q \\
q + 1 & \text{if } j = (l + 2)q. 
\end{cases} \]

Let \( \mathcal{E}_y \) be the indecomposable parity sheaf supported on the Schubert variety \( X_y \) with coefficients in \( \mathbb{F}_p \), extending the constant sheaf shifted by \( \dim(X_y) \). Our theorem is:

**Theorem 2.1.**

\[ \mathcal{E}_y \not\cong \mathcal{F} \quad \text{for } \tau \leq l - 3. \]

Here \( \mathcal{F} \) is the perverse truncation operator. Since \( l \geq 3 \), this implies that \( \mathcal{E}_y \) is not perverse.

### 3. Intersection Forms

If \( A \) is an indecomposable object in a Krull-Schmidt category and \( X \) is any object, write \( m(A, X) \) for the number of times \( A \) appears as a direct summand of \( X \).

Our main tool is the following result which computes the multiplicities of a direct summand via the rank of a bilinear form.

**Proposition 3.1.** [JMW, Proposition 3.2] Let \( k \) be a local ring. Let \( \pi : \tilde{Y} \to Y \) be a proper resolution of singularities. Let \( y \in Y \), and suppose that the fibre \( F = \pi^{-1}(y) \) is smooth. Write \( i \) for the inclusion of \( y \) in \( Y \). Let \( n \) be the dimension of \( \tilde{Y} \), \( d \) the dimension of \( F \) and \( m \) be an integer. Let \( B \) be the pairing

\[ H^{2d-n-m}(F) \times H^{2d-n+m}(F) \to H^{2d}(F) \]

given by \( B(\alpha, \beta) = \alpha \cup \beta \cup e \), where \( e \) is the Euler class of the normal bundle to \( F \) in \( Y \). Then

\[ m(i_*k[m], \pi_*k[n]) = \text{rank}(B). \]

**Proof.** By general results about multiplicities of indecomposable objects in Krull-Schmidt categories, the multiplicity \( m(i_*k[m], \pi_*k[n]) \) is equal to the rank of the pairing

\[ \text{Hom}(i_*k[m], \pi_*k_F[n]) \times \text{Hom}(\pi_*k_F[n], i_*k[m]) \to \text{End}(i_*k[m]) \cong k. \]  \( \text{(3.1)} \)

The following commutative diagram appears in the proof of [JMW, Lemma 3.4] and arises through applying base change and adjunctions. The pairing \( B' \) arises through composition with the canonical map from \( \omega_F[-n] \) to \( k_F[n] \) (which comes from applying the canonical natural transformation \( i^! \to i^* \) to \( k_F \)).
\[
\text{Hom}(kF[m], \omega F[-n]) \times \text{Hom}(kF[n], \omega F[m]) \xrightarrow{B'} \text{Hom}(kF[m], \omega F[m]) \\
\downarrow \quad \downarrow
\text{Hom}(i_*k[m], \pi_*kFY[n]) \times \text{Hom}(\pi_*kF[n], i_*k[m]) \longrightarrow \text{End}(i_*k[m])
\]

Since \( F \) is smooth, \( \omega F \cong kF[2d] \) and the canonical morphism from \( \omega F[-n] \) to \( kF[n] \) is identified with the Euler class \( e \). Therefore the pairing (3.1) is identified with the one stated in the proposition, completing the proof. \( \square \)

4. Geometry

Let \( x \) be the permutation

\[
\begin{pmatrix}
J & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & J & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & J & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & J
\end{pmatrix}
\]

where \( J \) is the \( q \times q \) antidiagonal matrix.

We compute the slice to the Schubert variety \( X_y \) at the point \( x \). Using the techniques discussed in [W], this is given by matrices of the form

\[
\begin{pmatrix}
0 & A_l \\
A_l & \vdots \\
A_l & A_2 \\
A_1 & A_2 \\
0 & B_1 & B_2 & \cdots & B_l & 0
\end{pmatrix}
\]

subject to the conditions

\( B_i J A_i = 0 \)

for all \( i \).

Let \( Y \) be this slice. Reimagine \( Y \) as the space of representations of the quiver

of dimension vector

\[
\begin{pmatrix}
q \\
q \\
q
\end{pmatrix}
\]
Let
\[ \tilde{Y} = \{(h, \ell_1, \ldots, \ell_l, \ell, A_1, \ldots, A_l, B_1', \ldots, B_l') \mid h \in Gr(q-1, q), \]
\[ \ell_1, \ldots, \ell_l, \ell \in Gr(1, q), A_i \in \text{Hom}(\mathbb{C}^q/h, \ell_i), B_i \in \text{Hom}(\mathbb{C}^q/\ell_i, \ell) \} \]

Then \( \tilde{Y} \) is the total space of a vector bundle \( E \) on \( Z := (\mathbb{P}^{q-1})^k + 2 \).

We have
\[ H^*(Z) \cong \mathbb{Z}[w]/(w^q) \otimes (\bigotimes_{i=1}^l \mathbb{Z}[a_i]/(a_i^q)) \otimes \mathbb{Z}[z]/(z^q) \]
where the variables \( w \) and \( z \) come from the factors of \( Z \) corresponding to the choices of \( h \) and \( \ell \) in \( \tilde{Y} \) respectively, while the \( a_i \) come from the choice of \( \ell_i \).

**Lemma 4.1.** The Euler class of \( E \) is given by
\[ e(E) = \prod_{i=1}^l \left( (a_i + w) \sum_{j=0}^{q-1} a_i^j z^{q-1-j} \right). \] (4.1)

**Proof.** The bundle \( E \) is naturally a direct sum of \( 2l \) vector bundles
\[ E \cong \bigoplus_{i=1}^l A_i \oplus \bigoplus_{i=1}^l B_i, \]
where the bundles \( A_i \) and \( B_i \) correspond to the choice of \( A_i \) and \( B_i \) in \( \tilde{Y} \). The bundle \( A_i \) is a line bundle with first Chern class \( a_i + w \).

On \( \mathbb{P}^{q-1} \), write \( L \) for the tautological line bundle, and \( T \) for the trivial vector bundle of rank \( q \). Then, restricted to the appropriate \( \mathbb{P}^{q-1} \times \mathbb{P}^{q-1} \), we have
\[ B_i \cong p_1^*(T/L)^* \otimes p_2^* L. \]

The total Chern class of \( (T/L)^* \) is given by
\[ c((T/L)^*) = \sum_{j=1}^\infty a_i^j. \]

The computation of \( e(B_i) \) proceeds via (4.2) below, whose proof follows easily using the splitting principle:

Let \( V \) be a rank \( n \) vector bundle and \( L \) be a line bundle. Then
\[ e(V \otimes L) = \sum_{i=0}^n c_i(V) c_1(L)^{n-i}. \] (4.2)

Using this, we obtain
\[ e(B_i) = \sum_{j=0}^{q-1} a_i^j z^{q-1-j}. \]

Together with the formula for the Euler class (=first Chern class) of \( A_i \), this implies the formula in the lemma. \( \Box \)
Motivated by Proposition 3.1, we consider the pairing
\[ \langle \cdot, \cdot \rangle : H^{2(q-k)}(Z) \times H^{2(q-2)}(Z) \to H^{2(k+2)}(Z) \]
given by
\[ \langle \sigma, \tau \rangle = \sigma \cup \tau \cup e(\mathcal{E}) \].

When computing the intersection pairing \( \langle \cdot, \cdot \rangle \) between two monomials in \( H^*(Z) \), one notices that it only depends on the power of \( w \) in their product. If this power is \( j \), then the pairing takes the value \( (lq - 1 - j) \), for this is the number of ways of choosing terms in the product formula (4.1) for \( e(\mathcal{E}) \) which produce the right power of \( w \), and once these choices are made, the rest are uniquely determined.

Deleting irrelevant rows, the matrix for our intersection form becomes
\[
M = \begin{pmatrix}
\binom{l}{2} & \binom{l}{3} & \cdots & l & 1 \\
1 & \binom{l}{2} & \binom{l}{3} & \cdots & l & 1 \\
1 & l & \binom{l}{2} & \cdots & \cdots & l & 1 \\
\cdots & \cdots & \cdots & l & 1 \\
1 & l & \binom{l}{2} & \cdots & l & 1 \\
1 & l & \binom{l}{2} & \cdots & l & 1
\end{pmatrix}
\] (4.3)

This matrix has \( q - 1 \) columns and \( q - l + 1 \) rows.

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**Lemma 5.1.** The matrix \( M \) from (4.3) has rank \( q - l + 1 \) over \( \mathbb{Q} \) and \( q - l \) over \( \mathbb{F}_p \).

**Proof.** Identify the rows of \( M \) with the sequence of polynomials

\[
(1 + x)^l - 1 \\
x(1 + x)^l \\
x^2(1 + x)^l \\
\vdots \\
x^{q-l-1}(1 + x)^l \\
x^{q-l}(1 + x)^l - x^q.
\]

If there is a linear dependence, then \( A + Bx^q \) is divisible by \((1 + x)^l\) for some constants \( A \) and \( B \). Over \( \mathbb{Q} \), this is impossible since \( A + Bx^q \) has distinct roots over \( \mathbb{C} \). This computes the rank over \( \mathbb{Q} \).

Over \( \mathbb{F}_p \), the polynomial \((1 + x)^l\) divides \( x^q + 1 \), which easily leads to a linear dependence amongst the rows of \( M \). It is obvious that the rank is at least \( q - l - 2 \), completing the proof in this case.

We now come to the proof of Theorem 2.1.
Proof. Let $\pi: \tilde{Y} \rightarrow Y$ be the canonical morphism between the spaces $\tilde{Y}$ and $Y$ defined in §4 and let $n$ be their common dimension. Decompose $\pi_*\mathbb{Z}_p[n]$ into indecomposables

$$\pi_*\mathbb{Z}_p[n] \cong \bigoplus_i \mathcal{E}_i^n.$$ 

Every fiber of $\pi$ is a product of projective spaces, so has no odd cohomology (thus $\pi$ is what is known as an even resolution). Therefore each $\mathcal{E}_i$, and hence each $\mathcal{E}_i \otimes \mathbb{F}_p$ is an indecomposable parity sheaf.

By Proposition 3.1 and Lemma 5.1, the multiplicity of $i_*\mathbb{Z}_p[l - 2]$ in $\pi_*\mathbb{Z}_p[n]$ is $q - l + 1$ when $k = \mathbb{Q}_p$ and $q - l$ when $k = \mathbb{F}_p$. Therefore there exists a unique $i$ such that

$$m(i_*\mathbb{Q}_p[l - 2], \mathcal{E}_i \otimes \mathbb{Q}_p) = 1.$$ 

This setup is symmetric under the action of the symmetric group $S_l$. So by uniqueness of $i$, $\mathcal{E}_i$ is $S_l$-invariant.

There are only two $S_l$-invariant intermediate strata in $Y$. One is where $A_1 = \ldots = A_l = 0$ and the other is $B_1 = \ldots = B_l = 0$. They can be treated similarly.

In each case the stratum has an even resolution by the total space of a vector bundle $\mathcal{F}$ over $\mathbb{P}^{q-1}$, where the zero section is the fibre over $0 \in Y$, and $e(\mathcal{F})$ is a power of $c_1(\mathcal{O}(1))$.

Thus in computing the intersection form, once irrelevant rows are deleted, one is left with the antidiagonal matrix $J$, which has the same rank over $\mathbb{Q}$ and $\mathbb{F}_p$.

By a similar argument to the decomposition of $\pi_*\mathbb{Z}_p[n]$, the parity sheaves supported on these strata cannot be the $\mathcal{E}_i$ above.

Therefore $\mathcal{E}_i$ is the parity extension of the shifted constant sheaf on $Y$. As the stalk of $\mathcal{E}_i$ at $0$ is free over $\mathbb{Z}_p$ and satisfies a parity vanishing property, it is $\mathbb{Z}_p$ in degree $l - 2$. Therefore $\mathcal{E}_i \otimes \mathbb{F}_p$ has nonzero stalk cohomology at $0$ in degree $l - 2$. Thus

$$\mathcal{E}_i \otimes \mathbb{F}_p \not\cong p_{\tau \leq l-3}(\mathcal{E}_i \otimes \mathbb{F}_p).$$

Since $\mathcal{E}_i \otimes \mathbb{F}_p$ is the restriction of $\mathcal{E}_y$ to the slice $Y$, the analogous statement holds for $\mathcal{E}_y$, completing the proof. □

References


[MW] Peter J. McNamara and Geordie Williamson, Tame Torsion. in preparation.


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