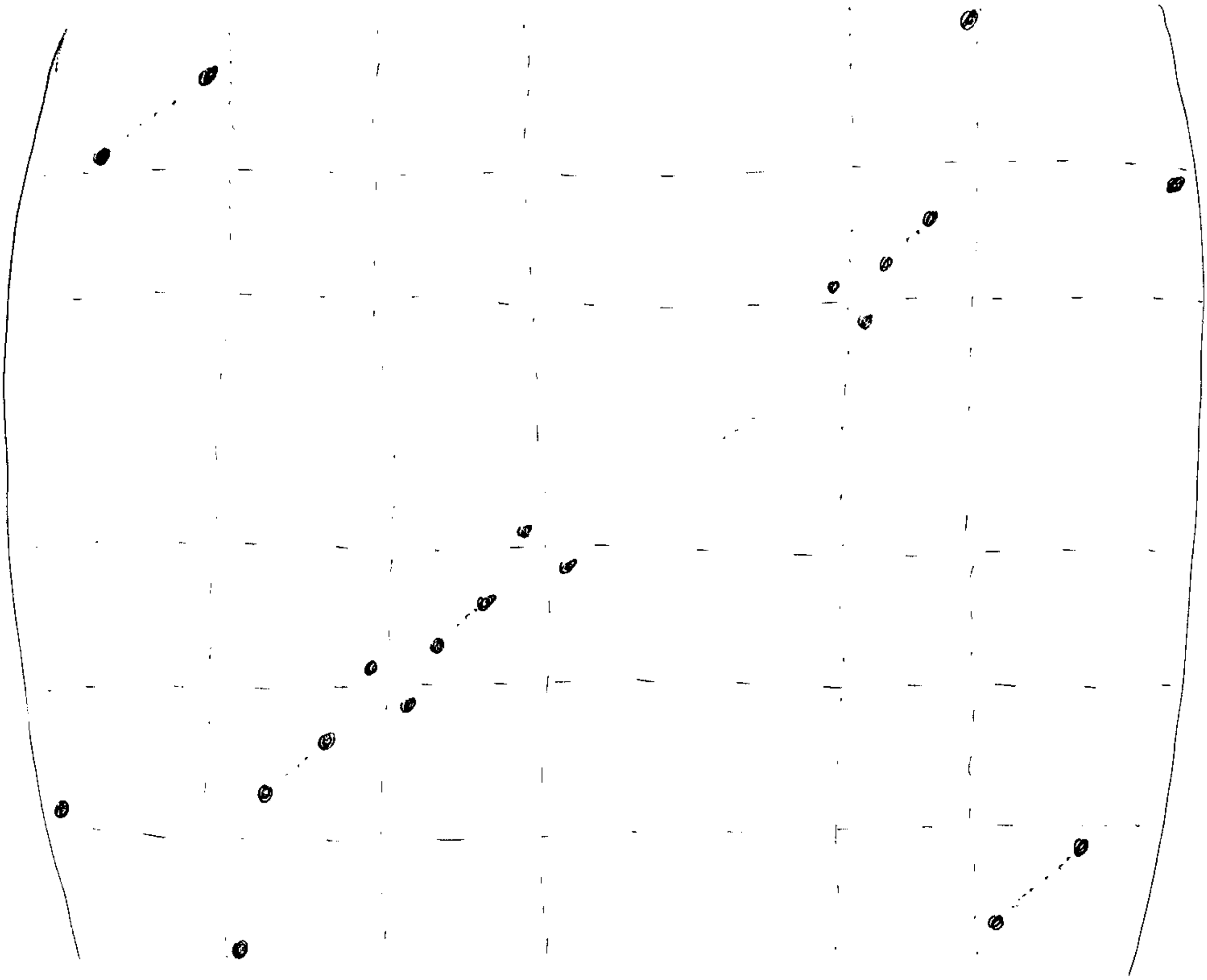
Let $q \geq k \geq 3$ be positive integers, p a prime, ~~divisor of~~ $q = p^d$.

Let y be the following $q(k+2) \times q(k+2)$ permutation matrix



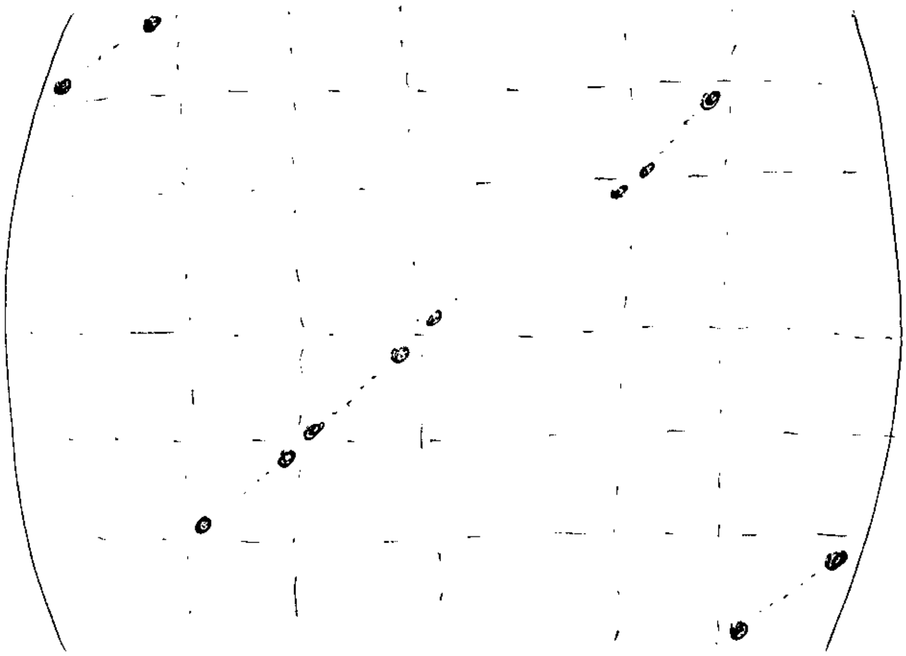
This matrix is written as a $(k+2) \times (k+2)$ block matrix, where each block is of size $q \times q$. Dots indicate the location of 1s.

Let $\mathcal{P}_y$ be the indecomposable parity sheaf supported on the Schubert variety through y, ~~in char~~ with coefficients in $\mathbb{F}_p$.

We will show

$$ {}^{p}H^{k-2}(\mathcal{P}_y) \neq 0 . $$

Let x be the permutation



We compute the slice to the Schubert variety X_x at x inside X_y.
It is given by matrices of the form

$$\begin{pmatrix} A_k & & & & \\ \vdots & & & & \\ A_2 & & & & \\ A_1 & & & & \\ & B_1 & B_2 & \cdots & B_k \end{pmatrix}$$

subject to the conditions:

$$\operatorname{rank}\begin{pmatrix} A_k \\ \vdots \\ A_2 \\ A_1 \end{pmatrix} \le 1, \quad \operatorname{rank}\begin{pmatrix} B_1 & B_2 & \cdots & B_k \end{pmatrix} \le 1, \quad B_i J A_i = 0 \text{ for } 1 \le i \le k,$$

$$\text{where } J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Let X be this slice. It will suffice to show the parity sheaf on X has non-vanishing $(k-2)$-th perverse cohomology.

Reimagine X as a space of representations of the quiver

$$\begin{tikzcd}[row sep=small]
 & \bullet \arrow[ddr, "B_1 J"] & \\
 & \bullet \arrow[dr, "B_2 J"] & \\
\bullet \arrow[uur, "A_1"] \arrow[ur, "A_2"] \arrow[ddr, "A_k"'] & \vdots & \bullet \\
 & & \\
 & \bullet \arrow[uur, "B_k J"'] &
\end{tikzcd}$$

of dimension vector

$$\begin{tikzcd}[row sep=small]
 & q \arrow[ddr] & \\
 & q \arrow[dr] & \\
q \arrow[uur] \arrow[ur] \arrow[ddr] & \vdots & q \\
 & & \\
 & q \arrow[uur] &
\end{tikzcd}$$

Let

$$\widetilde{X} = \left\{ (h, l_1, \ldots, l_k, l, A_1, \ldots, A_k, B'_1, \ldots, B'_k) \;\middle|\; \begin{array}{c} h \in Gr(q-1, q) \\ l_1, \ldots, l_k, l \in Gr(1, q) \\ A_i \in Hom(\mathbb{C}^q/h, l_i) \\ B'_i \in Hom(\mathbb{C}^q/l_i, l) \end{array} \right\} \qquad (B'_i = B_i J)$$

Then $\widetilde{X}$ is the total space of a vector bundle $\mathcal{E}$ on $Z := (\mathbb{P}^{q-1})^{k+2}$

The natural map $\pi : \widetilde{X} \longrightarrow X$ is an even resolution of singularities of X.

Consider the pairing $\langle \cdot, \cdot \rangle : H^{2(q-k)}(Z) \times H^{2(q-2)}(Z) \longrightarrow H^{2(k+2)(q-1)}(Z)$

$$\langle \sigma, \tau \rangle = \sigma \cup \tau \cup e(\mathcal{E}).$$

The rank of this pairing determines the multiplicity of the skyscraper sheaf $\delta_0[k-2]$ in the parity sheaf $\pi_* \underline{\mathbb{F}}_{\widetilde{X}}[(k+2)(q-1)]$.

There is

$$H^*(Z) \cong \mathbb{Z}[\omega]/(\omega^q) \otimes \left(\bigotimes_{i=1}^{k} \mathbb{Z}[a_i]/(a_i^q) \right) \otimes \mathbb{Z}[z]/(z^q)$$

and

$$e(\mathcal{E}) = \prod_{i=1}^{k} (a_i + \omega) \sum_{j=0}^{q-1} a_i^j z^{q-1-j}$$

When computing the pairing $\langle \cdot , \cdot \rangle$ between two monomials in $H^*(Z)$, one notices that it depends on the power of ω in their product. If this power is j, then the pairing takes the value $\binom{k}{q-1-j}$, for this is the number of choices of choosing terms in the product for $e(\mathcal{E})$ which produce the right power of ω, and these choices uniquely determine the others.

Deleting irrelevant rows, the matrix for our intersection form becomes

$$\begin{pmatrix} k & \binom{k}{2} & \binom{k}{3} & \cdots & k & 1 & & & & 0 \\ 1 & k & \binom{k}{2} & \binom{k}{3} & \cdots & k & 1 & & & \\ & 1 & k & \binom{k}{2} & & & \ddots & \ddots & & \\ & & & \ddots & \ddots & & & k & 1 & \\ 0 & & & 1 & k & \binom{k}{2} & \cdots & & k & 1 \\ & & & & 1 & k & \binom{k}{2} & \cdots & & k \end{pmatrix}$$

This matrix has $q-1$ columns and $q-k+1$ rows.

<u>Lemma:</u> The rank of this matrix is $q-k+1$ over $\mathbb{Q}$, and $q-k$ over $\mathbb{F}_p$.

<u>Pf:</u> Identify the rows of this matrix with the polynomials

$$\begin{gathered} (1+x)^k - 1 \\ x(1+x)^k \\ x^2(1+x)^k \\ \vdots \\ x^{q-k}(1+x)^k - x^q . \end{gathered}$$

If there is a linear dependence, then $A + Bx^q$ is divisible by $(1+x)^k$ for some A, B, but this is impossible over $\mathbb{Q}$ as $A + Bx^q$ has distinct roots in $\mathbb{C}$.

Over $\mathbb{F}_p$, $(1+x)^k \mid x^q + 1$ which easily leads to a linear dependance. $\square$

Since the rank drops by 1, there is thus a contribution to P_H^{k-2} unless this is absorbed by an intermediate strata.

The resolution $\widetilde{X} \longrightarrow X$ is S_k-equivariant. Since the rank only drops by one, if any intermediate stratum is responsible, it must be S_k-invariant.

There are only two S_k-intermediate strata. One is where $A_1 = A_2 = \cdots = A_k = 0$ and the other is where $B_1 = B_2 = \cdots = B_k = 0$. They can be treated similarly.

In each case the stratum has an even resolution by the total space of a vector bundle $\mathcal{F}$ over $\mathbb{P}^{q-1}$, where the zero section is the fibre over $0 \in X$, and $e(\mathcal{F})$ is a power of $c_1(\mathcal{O}(1))$.

Thus in computing the intersection form, once irrelevant rows are removed, we get the matrix

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

which has the same rank over $\mathbb{Q}$ and over $\mathbb{F}_p$. Thus these intermediate strata cannot be resposible for the drop in rank computed earlier, completing the proof.