NOTES ON R-MATRICES FOR QUIVER HECKE ALGEBRAS

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Notation: $k$ is the ground field. If the Cartan datum is symmetric, any $k$ will do. If there are $i, j \in I$ with $i \cdot i > j \cdot j$, let $p$ be a prime dividing $i \cdot i/j \cdot j$ and assume the characteristic of $k$ is $p$. We need a single possible choice for all pairs $i, j \in I$ so sometimes this is not possible. However there exists such a prime $p$ for all irreducible Cartan data of finite or affine type.

Given an integer $m$, define $\iota_m: S_n \to S_m$ by $\iota_m(w)(im - j) = w(i)m - j$ for $1 \leq i \leq n, 0 \leq j < m$.

The definition of the Quiver Hecke Algebras requires some polynomials $Q_{i,j}(u,v)$ for any $i, j \in I$. We make the assumption that $Q_{i,j}(u + z^{i/2}v + z^{j/2}) = Q(u,v)$. This includes all Quiver Hecke algebras coming from geometry. In the nonsymmetric case, this is where we need our assumptions about the characteristic of the field $k$.

Therefore there is a homomorphism $\psi_z: R(\nu) \to k[z] \otimes R(\nu)$ defined by

$$
\psi_z(e_i) = e_i \\
\psi_z(y_j e_i) = (y_j + z^{i/2}) \\
\psi_z(\tau_j) = \tau_j.
$$

The element $z$ is placed in degree two.

The intertwiners $\varphi_{ij}$ are defined as in [KKK, §1.3] and [KKK, Lemma 1.3.1] holds. So we can define

$$R_{M,N}: M \circ N \to q^1 N \circ M$$

as in [KKK] where $i = (\beta, \gamma) - 2(\beta, \gamma)_n$ if $M$ is a $R(\beta)$-module and $N$ is a $R(\gamma)$-module.

For a $R(\nu)$-module $M$, we define $M_z$ to be the $R(\nu)$-module $k[z] \otimes M$ with the action of $R(\nu)$ twisted by $\psi_z$.

Now consider two parameters $z$ and $w$, and the morphism

$$R_{M_z,N_w}: M_z \circ N_w \to q^1 N_w \circ M_z$$

Let $I_{M,N} = \{ f \in k[z,w] | f(N \circ M) \subset \text{im}(R_{M,N}) \}$. This is an ideal of $k[z,w]$. In [KKK] it is proved that $I_{M,N} = ((z - w)^s)$ for some $s \in \mathbb{N}$ in symmetric type. I do not know any example where $I_{M,N}$ is not principal.

Let $(\lambda, \mu)$ be a point of $\mathbb{R}^2 \setminus \{(0,0)\}$. We make the substitution $z = \lambda t$ and $w = \mu t$ to create a morphism

$$R^t_{M,N}(\lambda, \mu): k[t] \otimes M \circ N \to q^1 k[t] \otimes N \circ M.$$ 

Let $s$ be the largest integer such that the image of $s$ lies in $t^s q^1 k[t] \otimes N \circ M$. This exists by the same argument as in [KKK] for a generic choice of $\lambda$ and $\mu$.

Define the $R$-matrix $r_{M,N}(\lambda, \mu): M \circ N \to q^{1-2s} N \circ M$ by

$$r_{M,N}(\lambda, \mu) = (t^{-s} R^t_{M,N}(\lambda, \mu)) |_{t=0}. $$
We’ve defined a rational map from $\mathbb{P}^1$ to $\mathbb{P} \text{Hom}(M \circ N, N \circ M)$. Since the latter space is proper and the former space is a smooth curve, this extends to a morphism from $\mathbb{P}^1$ to $\mathbb{P} \text{Hom}(M \circ N, N \circ M)$. Therefore for any $[\lambda : \mu] \in \mathbb{P}^1$, we have defined a nonzero morphism from $M \circ N$ to $N \circ M$, well defined up to multiplication by a nonzero scalar.

It could be convenient to take $[\lambda : \mu] = [1 : 1]$. If we want to talk about a canonical $R$-matrix when we do not know that $I_{M,N}$ is principal then we will use $r_{M,N}(1,1)$.

**Lemma 0.1.** Let $L$, $M$ and $N$ be three modules. Then there is a scalar $c$ such that
\[
(id_M \circ r_{L,N})(r_{L,M} \circ id_N) = cr_{L,M \circ N}
\]

Since the $r$-morphisms are never zero, if either of $r_{L,M}$ or $r_{L,N}$ is an isomorphism, then the constant $c$ is nonzero.

The version with parameters is
\[
(id_M \circ r_{L,N}(\lambda, \mu))(r_{L,M}(\lambda, \mu) \circ id_N) = cr_{L,M \circ N}(\lambda, \mu).
\]

**Lemma 0.2.** Let $L$, $M$ and $N$ be three modules. Then
\[
(r_{M,N} \circ id_L)(id_M \circ r_{L,N})(r_{L,M} \circ id_N) = (id_N \circ r_{L,M})(r_{L,N} \circ id_M)(id_L \circ r_{M,N})
\]

The version with parameters is
\[
(r_{M,N}(\mu, \nu) \circ id_L(id_M \circ r_{L,N}(\lambda, \nu))(r_{L,M}(\lambda, \mu) \circ id_N) = (id_N \circ r_{L,M}(\lambda, \mu))(r_{L,N}(\lambda, \nu) \circ id_M)(id_L \circ r_{M,N}(\mu, \nu)).
\]

The following key result is [KKKO, Theorem 3.2].

**Theorem 0.1.** Let $X$ and $Y$ be simple representations at least one of which is real. Then $X \circ Y$ has a simple socle and a simple head. The socle of $X \circ Y$ is the image of $r_{Y,X}$.

**Theorem 0.2.** Let $X$, $Y$ and $Z$ be modules such that $X \circ X$, $X \circ Y$, $Y \circ Z$ and $Z \circ X$ are all irreducible. Then $X \circ Y \circ Z$ is irreducible.

**Proof.** The hypotheses that $X \circ Y$ and $X \circ Z$ are irreducible imply that $r_{X,Y}$ and $r_{X,Z}$ are isomorphisms. By Lemma 0.1, $r_{X,Y \circ Z}$ is an isomorphism.

Since $X$ is real and $Y \circ Z$ is irreducible, we can apply [KKKO, Theorem 3.2] to conclude that $X \circ Y \circ Z$ is irreducible, as required.

**Remark 0.3.** This is the Quiver Hecke analogue of a result of Hernandez [H] with an extra assumption that $X$ is real which would ideally be removed.

**Theorem 0.3.** Let $\alpha$ be a real root and $L$ a cuspidal representation of $R(\alpha)$ for some convex order. Then $L$ is a real module.

**Proof.** In finite type, this is [M1, Lemma 3.4]. The general case is by an unpublished argument explained to me by Ben Webster that is scheduled to appear in [M2].

**Remark 0.4.** There are other interesting examples of real representations. Eg in type $A_3$, the irreducible representation with character $[2132] + [312]$ is real (and is interesting since it categorifies a frozen cluster variable) but does not arise from the above constructions.

The following theorem is important for understanding imaginary semicuspidal modules, for example [KM, M2].

I will say that $A = \tau w_1 v_1 \otimes \cdots \otimes v_n$ plus lower order terms if the difference can be written as a sum of terms of the form $\tau w' v'_1 \otimes \cdots \otimes v'_n$ where $\ell(w') < \ell(w)$. Here $w, w'$ are taken to be minimal length coset representatives.
Theorem 0.4. Let $L$ be a cuspidal representation of $R(\delta)$, where $\delta \cdot \delta = 0$. The $R$-matrix $r_{L,L}$ (appropriately normalised) induces $n-1$ endomorphisms of $L^\otimes n$, denoted $r_1, \ldots, r_{n-1}$. There is an isomorphism from $k[S_n]$ to $\text{End}(L^\otimes n)$ sending the simple reflection $s_i$ to $r_i$.

Proof. I’m going to prove more and include a proof of the fact that $r_{L,L}(\lambda, \mu)$ is independent of the parameters $\lambda$ and $\mu$, when it is appropriately normalised.

The standard Mackey argument shows that $\text{End}(L^\otimes n)$ has dimension at most $n!$ and is concentrated in degree zero. Therefore the integer $s$ in the construction of $r_{L,L}$ is equal to $(\delta, \delta)_n$.

Pick $v \in L$ such that $y_i v = 0$ for all $i$. Let $d = |\delta|$. Then $\varphi_{\iota\delta((12))} v \otimes v = Q(z,w) v \otimes v$ plus lower order terms for some (explicit) nonzero polynomial $Q(z,w)$ of degree $2(\delta, \delta)_n$. This computation shows that $r_{L,L}(\lambda, \mu)$ is generically not equal to a multiple of the identity.

Therefore $\dim \text{End}(L \circ L) = 2$. This implies that the Mackey filtration of $\text{Res}_{\delta, \delta} L \circ L$ splits. So there exists an element $\tau \in \text{End}(L \circ L)$ with $\tau(v \otimes v) = \tau_{\iota\delta((12))} v \otimes v$. As 1 and $\tau$ form a basis of $\text{End}(L \circ L)$, there exist $p, q \in k$ with $\tau^2 = p\tau + q$.

We normalise $r_{L,L}(\lambda, \mu)$ such that $r_{L,L}(\lambda, \mu) = \tau + A(\lambda, \mu)$ for some rational function $A$. This is possible from the observations two paragraphs prior.

By [KKK, Lemma 1.3.1(vi)], $r_{L,L}(\lambda, \mu) r_{L,L}(\mu, \lambda) = 1$. Therefore $A(\lambda, \mu) + A(\mu, \lambda) = -p$ and $A(\lambda, \mu) A(\mu, \lambda) = 1 - q$. Hence $A(\lambda, \mu)$ is a constant and so is $r_{L,L}(\lambda, \mu)$.

Thus we know the quadratic relations $r_{L,L}^2 = 1$. The braid relations follow from Lemma 0.2. Therefore there is a homomorphism $f$ from $k[S_n]$ to $\text{End}(L^\otimes n)$. It is injective since $f(w) v \otimes n = \tau_{\iota\delta(w)} v \otimes n$ plus lower order terms. It is surjective by a dimension count. □

References


