NOTES ON R-MATRICES FOR QUIVER HECKE ALGEBRAS

PETER J MCNAMARA

Notation: k is the ground field. If the Cartan datum is symmetric, any k will do. If there are $i, j \in I$ with $i \cdot i > j \cdot j$, let p be a prime dividing $i \cdot i / j \cdot j$ and assume the characteristic of k is p. We need a single possible choice for all pairs $i, j \in I$ so sometimes this is not possible. However there exists such a prime p for all irreducible Cartan data of finite or affine type.

Given an integer m, define $\iota_m : S_n \longrightarrow S_{mn}$ by $\iota_m(w)(im - j) = w(i)m - j$ for $1 \leq i \leq n$, $0 \leq j < m$.

The definition of the Quiver Hecke Algebras requires some polynomials $Q_{i,j}(u, v)$ for any $i, j \in I$. We make the assumption that $Q_{i,j}(u + z^{i \cdot i/2}, v + z^{j \cdot j/2}) = Q(u, v)$. This includes all Quiver Hecke algebras coming from geometry. In the nonsymmetric case, this is where we need our assumptions about the characteristic of the field k.

Therefore there is a homomorphism $\psi_z : R(\nu) \longrightarrow k[z] \otimes R(\nu)$ defined by

$$\psi_z(e_{\mathbf{i}}) = e_{\mathbf{i}}$$
$$\psi_z(y_j e_{\mathbf{i}}) = (y_j + z^{\mathbf{i}_j \cdot \mathbf{i}_j / 2})$$
$$\psi_z(\tau_j) = \tau_j.$$

The element z is placed in degree two.

The intertwiners φ_a are defined as in [KKK, §1.3] and [KKK, Lemma 1.3.1] holds. So we can define

$$R_{M,N} : M \circ N \longrightarrow q^i N \circ M$$

as in [KKK] where $i = (\beta, \gamma) - 2(\beta, \gamma)_n$ if M is a $R(\beta)$-module and N is a $R(\gamma)$-module.

For a $R(\nu)$-module M, we define M_z to be the $R(\nu)$-module $k[z] \otimes M$ with the action of $R(\nu)$ twisted by ψ_z.

Now consider two parameters z and w, and the morphism

$$R_{M_z, N_w} : M_z \circ N_w \longrightarrow q^i N_w \circ M_z$$

Let $I_{M,N} = \{f \in k[z, w] \mid f(N \circ M) \subset \operatorname{im}(R_{M_z,N_w})\}$. This is an ideal of $k[z, w]$. In [KKK] it is proved that $I_{M,N} = \langle (z - w)^s \rangle$ for some $s \in \mathbb{N}$ in symmetric type. I do not know any example where $I_{M,N}$ is not principal.

Let (λ, μ) be a point of $\mathbb{A}^2 \setminus \{(0, 0)\}$. We make the substitution $z = \lambda t$ and $w = \mu t$ to create a morphism

$$R^t_{M,N}(\lambda, \mu) : k[t] \otimes M \circ N \longrightarrow q^i k[t] \otimes N \circ M.$$

Let s be the largest integer such that the image of s lies in $t^s q^i k[t] \otimes N \circ M$. This exists by the same argument as in [KKK] for a generic choice of λ and μ.

Define the R-matrix $r_{M,N}(\lambda, \mu) : M \circ N \longrightarrow q^{i-2s} N \circ M$ by

$$r_{M,N}(\lambda, \mu) = \left(t^{-s} R^t_{M,N}(\lambda, \mu)\right)|_{t=0}.$$

We've defined a rational map from $\mathbb{P}^1$ to $\mathbb{P}\operatorname{Hom}(M \circ N, N \circ M)$. Since the latter space is proper and the former space is a smooth curve, this extends to a morphism from $\mathbb{P}^1$ to $\mathbb{P}\operatorname{Hom}(M \circ N, N \circ M)$. Therefore for any $[\lambda : \mu] \in \mathbb{P}^1$, we have defined a nonzero morphism from $M \circ N$ to $N \circ M$, well defined up to multiplication by a nonzero scalar.

It could be convenient to take $[\lambda : \mu] = [1 : 1]$. If we want to talk about a canonical R-matrix when we do not know that $I_{M,N}$ is principal then we will use $r_{M,N}(1,1)$.

Lemma 0.1. *Let L, M and N be three modules. Then there is a scalar c such that*

$$(id_M \circ r_{L,N})(r_{L,M} \circ id_N) = c r_{L,M\circ N}$$

Since the r-morphisms are never zero, if either of $r_{L,M}$ or $r_{L,N}$ is an isomorphism, then the constant c is nonzero.

The version with parameters is

$$(id_M \circ r_{L,N}(\lambda,\mu))(r_{L,M}(\lambda,\mu) \circ \mathrm{id}_N) = c r_{L,M\circ N}(\lambda,\mu).$$

Lemma 0.2. *Let L, M and N be three modules. Then*

$$(r_{M,N} \circ id_L)(id_M \circ r_{L,N})(r_{L,M} \circ id_N) = (id_N \circ r_{L,M})(r_{L,N} \circ id_M)(id_L \circ r_{M,N})$$

The version with parameters is

$$(r_{M,N}(\mu,\nu)\circ\mathrm{id}_L)(id_M\circ r_{L,N}(\lambda,\nu))(r_{L,M}(\lambda,\mu)\circ\mathrm{id}_N) = (\mathrm{id}_N\circ r_{L,M}(\lambda,\mu))(r_{L,N}(\lambda,\nu)\circ\mathrm{id}_M)(\mathrm{id}_L\circ r_{M,N}(\mu,\nu)).$$

The following key result is [KKKO, Theorem 3.2].

Theorem 0.1. *Let X and Y be simple representations at least one of which is real. Then $X \circ Y$ has a simple socle and a simple head. The socle of $X \circ Y$ is the image of $r_{Y,X}$.*

Theorem 0.2. *Let X, Y and Z be modules such that $X \circ X$, $X \circ Y$, $Y \circ Z$ and $Z \circ X$ are all irreducible. Then $X \circ Y \circ Z$ is irreducible.*

Proof. The hypotheses that $X \circ Y$ and $X \circ Z$ are irreducible imply that $r_{X,Y}$ and $r_{X,Z}$ are isomorphisms. By Lemma 0.1, $r_{X,Y\circ Z}$ is an isomorphism.

Since X is real and $Y \circ Z$ is irreducible, we can apply [KKKO, Theorem 3.2] to conclude that $X \circ Y \circ Z$ is irreducible, as required. □

Remark 0.3. This is the Quiver Hecke analogue of a result of Hernandez [H] with an extra assumption that X is real which would ideally be removed.

Theorem 0.3. *Let α be a real root and L a cuspidal representation of $R(\alpha)$ for some convex order. Then L is a real module.*

Proof. In finite type, this is [M1, Lemma 3.4]. The general case is by an unpublished argument explained to me by Ben Webster that is scheduled to appear in [M2]. □

Remark 0.4. There are other interesting examples of real representations. Eg in type A_3, the irreducible representation with character $[2132] + [2312]$ is real (and is interesting since it categorifies a frozen cluster variable) but does not arise from the above constructions.

The following theorem is important for understanding imaginary semicuspidal modules, for example [KM, M2].

I will say that $A = \tau_w v_1 \otimes \cdots \otimes v_n$ plus lower order terms if the difference can be written as a sum of terms of the form $\tau_{w'} v'_1 \otimes \cdots \otimes v'_n$ where $\ell(w') < \ell(w)$. Here w, w' are taken to be minimal length coset representatitves.

Theorem 0.4. *Let L be a cuspidal representation of $R(\delta)$, where $\delta \cdot \delta = 0$. The R-matrix $r_{L,L}$ (appropriately normalised) induces $n-1$ endomorphisms of $L^{\circ n}$, denoted $r_1, \ldots, r_{n-1}$. There is an isomorphism from $k[S_n]$ to End $(L^{\circ n})$ sending the simple reflection s_i to r_i.*

Proof. I'm going to prove more and include a proof of the fact that $r_{L,L}(\lambda, \mu)$ is independent of the parameters λ and μ, when it is appropriately normalised.

The standard Mackey argument shows that End $(L^{\circ n})$ has dimension at most $n!$ and is concentrated in degree zero. Therefore the integer s in the construction of $r_{L,L}$ is equal to $(\delta, \delta)_n$.

Pick $v \in L$ such that $y_i v = 0$ for all i. Let $d = |\delta|$. Then $\varphi_{\iota_d((12))} v \circ v = Q(z,w)\tau_w v \otimes v$ plus lower order terms for some (explicit) nonzero polynomial $Q(z,w)$ of degree $2(\delta,\delta)_n$. This computation shows that $r_{L,L}(\lambda,\mu)$ is generically not equal to a multiple of the identity.

Therefore $\dim$ End $(L \circ L) = 2$. This implies that the Mackey filtration of $\mathrm{Res}_{\delta,\delta} L \circ L$ splits. So there exists an element $\tau \in$ End $(L \circ L)$ with $\tau(v \otimes v) = \tau_{\iota_d((12))} v \otimes v$. As 1 and τ form a basis of End $(L \circ L)$, there exist $p, q \in k$ with $\tau^2 = p\tau + q$.

We normalise $r_{L,L}(\lambda,\mu)$ such that $r_{L,L}(\lambda,\mu) = \tau + A(\lambda,\mu)$ for some rational function A. This is possible from the observations two paragraphs prior.

By [KKK, Lemma 1.3.1(vi)], $r_{L,L}(\lambda,\mu) r_{L,L}(\mu,\lambda) = 1$. Therefore $A(\lambda,\mu) + A(\mu,\lambda) = -p$ and $A(\lambda,\mu)A(\mu,\lambda) = 1-q$. Hence $A(\lambda,\mu)$ is a constant and so is $r_{L,L}(\lambda,\mu)$.

Thus we know the quadratic relations $r_{L,L}^2 = 1$. The braid relations follow from Lemma 0.2. Therefore there is a homomorphism f from $k[S_n]$ to End $(L^{\circ n})$. It is injective since $f(w)v^{\otimes n} = \tau_{\iota_{|\delta|}(w)} v^{\otimes n}$ plus lower order terms. It is surjective by a dimension count. □

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