$U_q(\mathfrak{g})$

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References: [HK02] (Chapter 3)

In this talk, we will go through the definition and structure of $U_q(\mathfrak{g})$ and its modules. The gist of the results is that the structure theory of $U_q(\mathfrak{g})$ follows the familiar story from finite-dimensional semisimple and Kac-Moody Lie algebras, and the representation theory also follows the same lines via weights and integrability. The A_1-form and classical limit of $U_q(\mathfrak{g})$ will allow us to transfer results between the classical and the quantum. And finally, we will be able to prove semisimplicity of a certain category of representations using the results from this chapter.

1 Structure of $U_q(\mathfrak{g})$

First, we define $U_q(\mathfrak{g})$ by generators and relations from a Cartan matrix, and we discuss its structure as an algebra and as a Hopf algebra.

The q-notation used here is the same as in the first talk.

To begin the definition of $U_q(\mathfrak{g})$, we first need to recall the theory of Cartan matrices and some spaces associated with them.

Definition 1.1. *A matrix $A = (a_{ij})_{i,j=1}^n$ is called a symmetrisable generalised Cartan matrix if*

1. *There exists some diagonal matrix $D = \operatorname{diag}(s_i,\ i = 1, \dots, n)$ with $s_i \in \mathbb{Z}_{>0}$ such that DA is symmetric.*
2. *$a_{ii} = 2$ for all $i = 1, \dots, n$.*
3. *$a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$.*
4. *$a_{ij} = 0$ if and only if $a_{ji} = 0$.*

To the matrix A we now associate some spaces.

- *Let $\mathfrak{h} = \operatorname{span}\{h_i, d_s \mid i = 1, \dots, n,\ s = 1, \dots, n - \operatorname{rank} A\}$ so that $\mathfrak{h}$ is $(2n - \operatorname{rank} A)$-dimensional. Let $P^\vee = \operatorname{span}_{\mathbb{Z}}\{h_i, d_s\}$, called the Dual Weight Lattice.*
- *Let $P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}$, this is called the Weight Lattice.*
- *Set $\Pi^\vee = \{h_i,\ i = 1, \dots, n\}$, these are called the simple coroots.*

- *let* $\Pi = \{\alpha_i \mid i = 1, \dots, n\}$ *be a linearly independent subset of* $\mathfrak{h}^*$ *such that*

$$\alpha_j(h_i) = a_{ij}, \quad \alpha_j(d_s) \in \{0, 1\}.$$

Elements of Π *are called the simple roots.*

The collection $(A, \Pi, \Pi^\vee, P, P^\vee)$ *is called the Cartan datum associated to* A.

We may now define the quantum group $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$:

Definition 1.2. $U_q(\mathfrak{g})$ *is the associative algebra over* $\mathbb{F}(q)$*, with 1, generated by* e_i*,* f_i*, and* q^h*, for* $i = 1, \dots, n$ *and* $h \in P^\vee$ *with the following relations:*

1. $q^0 = 1$*,* $q^h q^{h'} = q^{h+h'}$
2. $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i$ *for* $h \in P^\vee$ *(note this power of* q *is then taken from the Cartan matrix)*
3. $q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$ *for* $h \in P^\vee$
4. $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ *for* $i, j = 1, \dots, n$
5. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ *for* $i \neq j$
6. $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ *for* $i \neq j$

where $q_i = q^{s_i}$ *and* $K_i = q^{s_i h_i}$.

We now note that all relations of $U_q(\mathfrak{g})$ are homogenous with respect to the grading defined by $\deg f_i = -\alpha_i$, $\deg q^h = 0$, and $\deg e_i = \alpha_i$. So $U_q(\mathfrak{g})$ has a root space decomposition:

$$U_q(\mathfrak{g}) = \bigoplus_{\alpha \in Q} (U_q)_\alpha.$$

where Q is the root lattice $\mathrm{span}_{\mathbb{Z}}\{\alpha_i\}$ and

$$(U_q)_\alpha) = \{u \in U_q(\mathfrak{g}) \mid q^h u q^{-h} = q^{\alpha(h)} u \ \forall h \in P^\vee\}.$$

The reason for this definition of root space is because $q^h u q^{-h}$ measures the degree of u in the power of q resulting. We can see this by taking products of the relations (2) and (3). Homogeneity of relations ensures that no elements in distinct root spaces are identified by the relations.

The other decomposition that we will note is the traignular decomposition. If U_q^+ (resp. U_q^-) is the subalgebra generated by e_i (resp. f_i), and U_q^0 is that generated by q^h for $h \in P^\vee$, then we have the decomposition as vector spaces

$$U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+.$$

Now we recall the definition of a Hopf algebra, the Hopf algebra structure of $U_q(\mathfrak{g})$, and note its use.

Definition 1.3. *A Hopf algebra H over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$ together with six maps: the multiplication $\mu : H \otimes H \to H$, the comultiplication $\Delta : H \to H \otimes H$, the unit $\iota : \mathbb{F} \to H$, the counit $\varepsilon : H \to \mathbb{F}$, and the antipode $S : H \to H$ satisfying the following conditions*

1. *H with μ and ι forms an associative algebra with identity $1 = \iota(1)$.*

2. *The following diagrams commute (these make H a coassociative coalgebra):*

$$\begin{tikzcd} H \arrow[r, "\cong"] \arrow[d, "\Delta"] & H \otimes \mathbb{F} \text{ (r. } \mathbb{F} \otimes H) \\ H \otimes H \arrow[ur, "\mathrm{id}\otimes\varepsilon \text{ (r. } \varepsilon\otimes\mathrm{id})"'] & \end{tikzcd} \qquad \begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "\Delta\otimes\mathrm{id}"'] \\ H \otimes H \arrow[u, "\mathrm{id}\otimes\Delta"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

3. *The algebra multiplication and unit are coalgebra homomorphisms, that is*

$$\begin{tikzcd} H \otimes H & H \arrow[l, "\Delta"'] \\ (H \otimes H) \otimes (H \otimes H) \arrow[u, "\mu\otimes\mu"] & H \otimes H \arrow[l, "(\mathrm{id}\otimes\sigma\otimes\mathrm{id})\circ(\Delta\otimes\Delta)"] \arrow[u, "\mu"'] \end{tikzcd} \qquad \begin{tikzcd} H \otimes H & H \arrow[l, "\Delta"'] \\ \mathbb{F} \otimes \mathbb{F} \arrow[u, "\iota\otimes\iota"] & \mathbb{F} \arrow[l, "1\otimes1\leftarrow1"] \arrow[u, "\iota"'] \end{tikzcd}$$

   and $\varepsilon_H \circ \mu(x \otimes y) = \varepsilon_H(x)\varepsilon_H(y)$, and $\varepsilon_H \circ \iota = \mathrm{id}_{\mathbb{F}}$.

4. *The comultiplication and the counit are algebra homomorphisms $H \to H \otimes H$ and $H \to \mathbb{F}$ respectively. That is (for the comultiplication) we have $\Delta \circ \iota = \iota \otimes \iota$ and*

$$\begin{tikzcd} H \otimes H \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & H \arrow[d, "\Delta"] \\ H \otimes H \otimes H \otimes H \arrow[r, "(\mu\otimes\mu)\circ(\mathrm{id}\otimes\sigma\otimes\mathrm{id})"'] & H \otimes H \end{tikzcd}$$

5. *The antipode S satisfies the following commutative diagram*

$$\begin{tikzcd} H \otimes H \arrow[r, "S\otimes\mathrm{id}\ (\mathrm{id}\otimes S)"] & H \otimes H \arrow[d, "\mu"] \\ H \arrow[u, "\Delta"] \arrow[r, "\iota\circ\varepsilon"] & H \end{tikzcd}$$

It turns out that the antipode is an antihomomorphism of the algebra H, ie. $S(xy) = S(y)S(x)$. We now give the Hopf algebra structure on $U_q(\mathfrak{g})$:

Proposition 1.1. *$U_q(\mathfrak{g})$ has a Hopf algebra structure defined by*

1. $\Delta(q^h) = q^h \otimes q^h$

2. $\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i$, $\Delta(f_i) = f_i \otimes 1 + K \otimes f_i$

3. $\varepsilon(q^h) = 1$, $\varepsilon(e_i) = \varepsilon(f_i) = 0$

4. $S(q^h) = q^{-h}$, $S(e_i) = -e_iK_i$, $S(f_i) = -K_i^{-1}f_i$.

for $h \in P^\vee$ *and* $i = 1, \dots, n$.

The reason for mentioning the Hopf algebra structure is to note that it allows us to define tensor products and duals of representations. In general, if H is a Hopf algebra and V is a representation thereof, then $V \otimes V$ is a representation of H under $x(v \otimes w) = \Delta(x)(v \otimes w)$ acting component-wise, and a dual representation may be defined on the vector space V^* by $(xf)(v) = f(S(x)v)$.

2 Rep. Theory of $U_q(\mathfrak{g})$

The purpose of this section is to recall the definitions in the representation theory of quantum groups, as well as to note the similarities to the theory of finite-dimensional semisimple, or Kac-Moody, Lie algebras. In later sections we will focus on the connection between the representation theory of $U_q(\mathfrak{g})$ and that of $U(\mathfrak{g})$.

Definition 2.1. *A* $U_q(\mathfrak{g})$*-module* V^q *is called a weight module if it admits a weight space decomposition:*

$$V^q = \bigoplus_{\mu \in P} V^q_\mu, \quad V^q_\mu = \{v \in V^q \mid q^h v = q^{\mu(h)} v \ \forall h \in p^\vee\}.$$

Nonzero elements of V^q_μ *are called weight vectors of weight* μ*. If* $e_i v = 0$ *for all* $i = 1, \dots, n$ *then* v *is called a maximal vector. If* $V^q_\mu \neq 0$*, then* μ *is called a weight of* V^q *and* V^q_μ *is the weight space of weight* μ*. The dimension of a weight space is called the multiplicity of* μ *in* V^q*.*

A weight module V^q *is called a highest weight module with highest weight* λ *if it is generated by a nonzero vector* $v_\lambda \in V^q_\lambda$ *for which* $e_i v_\lambda = 0$ *for all* $i = 1, \dots, n$*.*

Denote by $\mathrm{wt}(V^q)$ *the set of weights of* V^q*, and if* $\dim V^q_\mu < \infty$ *for all weights* μ *then define the character of* V^q *by*

$$\mathrm{ch} V^q = \sum_\mu \dim V^q_\mu e^\mu.$$

where e^μ *are formal symbols which we might later multiply in the natural way.*

We now consider the Verma modules of $U_q(\mathfrak{g})$ and note that they play the same role here as in the familiar cases.

Fix a weight $\lambda \in P$ and let $J^q(\lambda)$ denote the left ideal of $U_q(\mathfrak{g})$ (as an algebra) generated by the e_i and $q^h - q^{\lambda(h)}1$ for all $h \in P^\vee$. Then we may define the Verma module $M^q(\lambda) = U_q(\mathfrak{g})/J^q(\lambda)$, with the action given by left multiplication. Then $M^q(\lambda)$ is a highest weight module with highest weight λ and hw vector $v_\lambda = 1 + J^q(\lambda)$.

Proposition 2.1.

1. *As a* U_q^-*-module,* $M^q(\lambda)$ *is freely generated by* v_λ*.*
2. *Every highest weight* $U_q(\mathfrak{g})$*-module with highest weight* λ *is the image a module homomorphism from* $M^q(\lambda)$ *to that module.*
3. *The module* $M^q(\lambda)$ *has a unique maximal submodule, and hence yields a unique irreducible highest weight module with highest weight* λ*, called* $V^q(\lambda)$*.*

Definition 2.2. *The category $\mathcal{O}^q_{\text{int}}$ consists of weight modules V^q with finite-dimensional weight spaces for which*

1. *V^q has a weight space decomposition with finite-dimensional weight spaces.*
2. *There exist finitely many weights $\lambda_1, \ldots, \lambda_s \in P$ such that*

$$\text{wt}(V^q) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s), \quad D(\lambda) = \{\mu \in P \mid \mu \leq \lambda\}.$$

   with the standard partial order on weights: $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a nonnegative integer combination of simple roots.

3. *All e_i and f_i are locally nilpotent on V^q.*

Note that these are sometimes called "Type 1" weight modules because the eigenvalues in the weight spaces are all of the form $+q^{\lambda(h)}$ rather than $-q^{\lambda(h)}$. On the other hand, in the first talk we saw that the eigenvalues of the highest weight vector of a finite-dimensional $U_q(\mathfrak{sl}_2)$-module were of the form $\pm q^n$. So considering only Type 1 weight modules makes the results of that theorem unique.

We note that we have again a classification of irreducible highest-weight modules:

Proposition 2.2. *Let $V^q(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with hw $\lambda \in P$. Then $V^q(\lambda) \in \mathcal{O}^q_{\text{int}}$ if and only if $\lambda \in P^+$ (that is, λ is dominant integral in the usual sense).*

3 A_1 forms of $U_q(\mathfrak{g})$ and its modules

Now we turn to the theory which will allow us to draw the link to $U(\mathfrak{g})$. We will be able to use this, and the next section, to transfer some results from the $U(\mathfrak{g})$ to $U_q(\mathfrak{g})$. Since this is not actually our main concern, we will skip much of the detail but mention the utility.

First define the ring

$$A_1 = \{g/h \mid g, h \in \mathbb{F}[q],\ h(1) \neq 0\}.$$

Definition 3.1. *The A_1-form of $U_q(\mathfrak{g})$ is denoted by U_{A_1} and is the A_1-subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, q^h, and $\frac{q^h-1}{q-1}$, for $i = 1, \ldots, n$ and $h \in P^\vee$.*

Definition 3.2. *The A_1-form of the highest weight module V^q with highest weight $\lambda \in P$ and highest weight vector v_λ is defined to be the U_{A_1}-module $V_{A_1} = U_{A_1} v_\lambda$, with the action as it was in V^q.*

4 The classical limit

We modify the A_1-forms of our algebra and modules in a unique way to yield the classical objects.

The ring A_1 has a unique maximal ideal J_1, generated by $(q-1)$, so A_1/J_1 is a field. In fact it is isomorphic to $\mathbb{F}$ by the map $f(q) + J_1 \mapsto f(1)$.

We define two vector spaces over $\mathbb{F}$:

$$U_1 = (A_1/J_1) \otimes_{A_1} U_{A_1},$$

$$V^1 = (A_1/J_1) \otimes_{A_1} V_{A_1}.$$

We also have the following isomorphisms and natural maps

$$U_{A_1} \to U_{A_1}/J_1 U_{A_1} \cong U_1.$$

$$V_{A_1} \to V_{A_1}/J_1 V_{A_1} \cong V^1.$$

To see the isomorphism, consider the map $V_{A_1} \to V^1$ given by $V_{A_1} \mapsto 1 \otimes_{A_1} V_{A_1} \subset V^1$. The kernel is exactly $J_1 V_1$. Furthermore, V^1 is a U_1-module.

We use bar notation for the image of these maps, and refer to taking this quotient as taking the classical limit. We also write $\bar{h}$ for the classical limit of the element $\frac{q^h-1}{q-1}$.

The following theorem illustrates the use of the classical limit:

Theorem 4.1. *1. The elements $\bar{e_i}$, $\bar{f_i}$, and $\bar{h}$, $i = 1, \ldots, n$ and $h \in P^\vee$ satisfy the defining relations of $U(\mathfrak{g})$, the UAE of the Kac-Moody Lie algebra $\mathfrak{g}$ associated to A. Hence there is a surjective $\mathbb{F}$-algebra hom. $\psi : U(\mathfrak{g}) \to U_1$ (because these elements generate $U_q(\mathfrak{g})$, and the U_1-module V^1 becomes a $U(\mathfrak{g})$-module.*

2. *For each weight $\mu \in P$, $h \in P^\vee$, the element $\bar{h}$ acts on V^1_μ by scalar multiplication by $\mu(h)$. The space V^1_μ is defined as*

$$V^1_\mu = (A_1/J_1) \otimes_{A_1} (V_{A_1} \cap V^q_\mu).$$

It is essentially the image of our original quantum weight space under first taking the A_1-form and then the classical limit, and this part of the theorem says that it ends up a true weight space.

3. *As a $U(\mathfrak{g})$-module, V^1 is a hw module with hw λ and hw vector $\bar{v}_\lambda$.*

4. *For each weight $\mu \in P$, $\dim_{\mathbb{F}} V^1_\mu = \dim_{\mathbb{F}(q)} V^q_\mu$. That is, the dimension of weight spaces are preserved.*

5. *We have* $\mathrm{ch} V^1 = \mathrm{ch} V^q$

Now we can transfer just enough results to build a concrete connection between the representation theories of $U_q(\mathfrak{g})$ and $U(\mathfrak{g})$.

Theorem 4.2. *If $\lambda \in P^+$ and V^q is the canonical hw module $V^q(\lambda)$, then V^1 is isomorphic to the irrep $V(\lambda)$ of $U(\mathfrak{g})$.*

Proof. Let v_λ be the highest weight vector of V^q. By the previous theorems, V^1 is a highest weight $U(\mathfrak{g})$-module with hw λ, hwv $\bar{v}_\lambda$, satisfying

$$f_i^{\lambda(h_i)+1} \bar{v}_\lambda = \bar{f}_i^{\lambda(h_i)+1} \bar{v}_\lambda = 0.$$

The first expression is the action of $U(\mathfrak{g})$ on V^1. This is defined in terms of $U_1(\mathfrak{g})$ acting on V^1, which in turn is defined by the action of the quantum group on V^q. It is the case that in the standard irrep, $f_i^{\lambda(h_i)+1}v_\lambda = 0$. Hence this expression is zero.

Furthermore, in the classical case, having this identity on a highest weight module is enough to show that it is precisely $V(\lambda)$. So the classical limit of $V^q(\lambda)$ is exactly $V(\lambda)$. □

Corollary 4.1. *Let $\lambda \in P^+$ and V^q be a hw module with hwv v_λ, hw λ. If $f_i^{\lambda(h_i)+1}v_\lambda = 0$, then $V^q \cong V^q(\lambda)$.*

This is analogous to a theorem in the classical case, and while the other direction can be shown directly for quantum groups, our new machinery allows us to quickly prove this direction.

Proof. By the previous theorems V^1 is a hw module with hwv v_λ,weight λ. And indeed

$$f_i^{\lambda(h_i)+1}\bar{v}_\lambda = \bar{f}_i^{\lambda(h_i)+1}\bar{v}_\lambda = 0.$$

Note that this is a relation in the $U(\mathfrak{g})$-module V^1. As previously noted, this relation implies that $V^1 \cong V(\lambda)$, the canonical irrep.. To transfer this back to a statement about quantum-group reps we use the theorem on characters:

$$\mathrm{ch}V^q = \mathrm{ch}V^1 = \mathrm{ch}V(\lambda) = \mathrm{ch}V^q(\lambda)$$

where the final equality is by the previous theorem.

By the fact that V^q and $V^q(\lambda)$ are both quotients of the Verma module $M^q(\lambda)$, but that $V^q(\lambda)$ is the quotient by a maximal ideal, there exists a surjective weight-preserving hom. $V^q \to V^q(\lambda)$. Since the characters agree, this is an isomorphism. □

Similarly we could prove:

Corollary 4.2.

1. *If V^q is a hw $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ with hw $\lambda \in P$, then $\lambda \in P^+$ and $V^q \cong V^q(\lambda)$.*
2. *Every irreducible $U_q(\mathfrak{g})$-module in the category $\mathcal{O}^q_{\mathrm{int}}$ is isomorphic to $V^q(\lambda)$ for some $\lambda \in P^+$.*

I'll just note now two important theorems which we might have expected, and then we will move on to semisimplicity of $\mathcal{O}^q_{\mathrm{int}}$

Theorem 4.3.

1. *The classical limit U_1 of $U_q(\mathfrak{g})$ has a Hopf algebra structure making it isomorphic to $U(\mathfrak{g})$*
2. *The classical limit of the quantum group Verma module $M^q(\lambda)$ is isomorphic to $M(\lambda)$, the $U(\mathfrak{g})$-module.*

5 Semisimplicity of $\mathcal{O}^q_{\text{int}}$

We show two important properties of $\mathcal{O}^q_{\text{int}}$ — that it is closed under taking tensor products, and that it is semisimple.

Proposition 5.1. *Let $V^q, W^q \in \mathcal{O}^q_{\text{int}}$. Then $V^q \otimes W^q \in \mathcal{O}^q_{\text{int}}$.*

Note that the weight space decomposition is

$$V^q \otimes W^q = \sum_{\lambda,\ \mu} V^q_\lambda \otimes W^q_\mu = \sum_{\lambda+\mu} (V^q \otimes W^q)_{\lambda+\mu},$$

where λ are weights of V^q and μ are weights of W^q, and we have $V^q_\lambda \otimes W^q_\mu = (V^q \otimes W^q)_{\lambda+\mu}$.

We now can prove the semisimplicity of $\mathcal{O}^q_{\text{int}}$. First we must define some dual modules:

Definition 5.1. *Suppose $V \in \mathcal{O}^q_{\text{int}}$ such that $V = \bigoplus_\mu V_\mu$. Define two dual modules, with actions, by*

$$V^* = \bigoplus_\mu V^*_\mu; \quad (x\phi)(v) = \phi(S(x)v)$$

$$V' = \bigoplus_\mu V^*_\mu; \quad (x\phi)(v) = \phi(S^{-1}(x)v).$$

It turns out that

Lemma 5.1. *1. There are isomorphisms $(V^*)' \cong V \cong (V')^*$.*

2. V^*_μ is a weight space of V^* with weight $-\mu$

3. V^* is integrable and $\mathrm{wt}(V^*) \subset \cup(-\lambda_j + Q_+)$ where $\mathrm{wt}(V) \subset \cup(\lambda_j - Q_+)$ (both finite unions)

Suppose that $V \in \mathcal{O}^q_{\text{int}}$. Then we may choose a maximal weight λ and nonzero $v_\lambda \in V_\lambda$, and set $L = U_q(\mathfrak{g})v_\lambda$. Then $L \cong V^q(\lambda)$ (by virtue of being a highest weight module with highest weight $\lambda \in P$).

Let $v^*_\lambda \in V^*_\lambda$ be defined by $v^*_\lambda(v_\lambda) = 1$ and $v^*_\lambda(V_\mu) = 0$ for $\mu \neq \lambda$.

Let $\bar{L}$ be the module generated by v^*_λ. Then it turns out that $\bar{L} \cong (V^q(\lambda))^*$, the irred. lowest weight module with lowest weight $-\lambda$, lowest weight vector v^*_λ.

We now identify an irreducible component of V.

Lemma 5.2. *Let $V \in \mathcal{O}^q_{\text{int}}$ and let L be the submodule of V generated by v_λ of a maximal weight. Then as a direct sum of modules,*

$$V \cong L \oplus V/L.$$

Proof. We show that in the ses

$$0 \to L \to V \to V/L \to 0.$$

there is an left inverse to the injection $L \to V$ (ie ψ st $\psi \circ \iota = \mathrm{id}_L$). Now since there is an injection $\bar{L} \to V^*$, taking the "dash" dual gives a map $(V^*)' \to (\bar{L})'$. But there is a map $L \to V$, and $(V^*)' \cong V$, so

$$L \xrightarrow{\iota} V \xrightarrow{\varphi} (\bar{L})' \ .$$

Since v_λ is not zeroed by the maps, their composition is an isomorphism (by Schurs lemma, it's either 0 or an isomorphism, but it's not zero). Composing the inverse of the isomorphism with φ we get a left inverse of ι. Hence the sequence splits (this is a theorem of short exact sequences) so

$$V \cong L \oplus V/L.$$

□

Now we can prove the theorem of semisimplicity

Theorem 5.1. *Every $V \in \mathcal{O}^q_{\mathrm{int}}$ is a direct sum of $V^q(\lambda)$, $\lambda \in P^+$.*

Proof. Let

$$F = \bigoplus_{\lambda \text{ dom.int.}} V_\lambda.$$

Then F is a finite-dimensional $U_q^{\geq 0}$-module, due to the boundedness from above of the weights of V. Let $V_F = U_q(\mathfrak{g})F$.

We can choose a maximal weight vector of F and apply the lemma to get

$$V_F = L \oplus L_1,$$

with L an irreducible highest weight module with dominant integral highest weight. And $L_1 = V_F/L$, which is generated by $F/(F \cap L)$, which has lower dimension than F. Choosing again a maximal weight vector from $F/(F \cap L)$, we repeat the process to decompose V_F as a direct sum of irreducibles. This process terminates because the generating sets are finite-dimensional and of smaller dimension each time. So V_F is a finite direct sum of irreducibles.

Now we show $V_F = V$. Consider $V/V_F \in \mathcal{O}^q_{\mathrm{int}}$. If this is nonzero, it has a maximal weight vector v_λ , and hence the irrep $V^q(\lambda)$ appears:

$$V^q(\lambda) \to V/V_F$$

but $V^q(\lambda)$, being irreducible, has λ dominant integral, so $v_\lambda \in V_F$ and so indeed $v_\lambda = 0$ in V/V_F (by definition of V_F). Hence $V = V_F$. □