SEMINAR ON QUANTUM FIELD THEORY
THE UNIVERSITY OF QUEENSLAND – SEMESTER I, 2018

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The aim of this seminar is to understand certain aspects of quantum mechanics from a mathematical point of view.

For Semester I, 2018, we will focus on the definition of a crystal basis and how it can be used to help us understand the representation theory of Lie algebras. In particular, crystal bases gives us a combinatorial interpretation of algebraic objects and has a number of applications to mathematical physics.

Some good general references:

- Book by Bump and Schilling [BS17] – builds up highest weight crystals from combinatorics; primarily restricted to finite types.
- Book by Hong and Kang [HK02] – starts from representation theory of quantum groups; constructs highest weight crystal theory; does not discuss $B(\infty)$.
- The original Kashiwara papers [Kas90, Kas91] – the grand-loop argument is given in [Kas91].
- The SageMath documentation:
  - Main reference entry point: `http://doc.sagemath.org/html/en/reference/index.html`; specifically, see the crystals documentation.
  - Thematic tutorials entry point: `http://doc.sagemath.org/html/en/thematic_tutorials/index.html`; specifically, see the one on Lie theory.

Talk 0: Introduction (Travis Scrimshaw)

Travis will give an overview of the topic of the seminar. We will also discuss the distribution of the talks.

Talk 1: The quantum group $U_q(\mathfrak{sl}_2)$ and bases of tensor products (Mariel)

The representation theory of $U_q(\mathfrak{sl}_2)$ forms the building block of everything we will do for the rest of the semester. We construct the finite-dimensional representations and their tensor products. We discuss a better basis that is better behaved with respect to the decomposition into irreducibles. See [HK02, Ch. 1,§4.3,§4.4] or [Kas02, Ch. 1] (in French).

Talk 2: Classical representation theory of $\mathfrak{g}$ (Naso)

We discuss the representation theory for a simple Lie algebra $\mathfrak{g}$, including the Weyl group action, tensor products, the universal enveloping algebra, and the adjoint representation. Other topics may include Verma modules, and constructing highest weight irreducible representations. See [HK02, Ch. 2]. This will give a more general picture of what crystal bases are meant to encode.

Talk 3: The quantum group $U_q(\mathfrak{g})$ (Joe)

We recall the definition of the (Drinfel'd–Jimbo) quantum group and some basic properties of its representation theory. We describe how the Hopf algebra structure applies to its representation theory. We give a construction of Verma modules and highest weight irreducible representations. See [HK02, Ch. 3].

Talk 4: Crystal bases: definition and examples (Mariel)

This is the main definition of the semester: a crystal basis [HK02, Ch. 4][Kas91]. We construct the crystal bases for many examples by using minuscule posets and the adjoint representation (this is the finite-type version of [OS18, §3.1.1,§3.1.2]).

Talk 5: Abstract crystals and the tensor product rule (Sam)

We give an axiomatic approach to crystals based on the properties of crystal bases for irreducible representations [HK02, §4.5]. (Most papers on crystals have this definition or some restricted variant.) Furthermore, we show the tensor product rule extends to irreducible representations [HK02, Thm. 4.4.1,Thm. 4.4.3] and give the signature rule [BS17, §2.4].

Talk 5: Perfect bases (Aid.)

We introduce the notion of a perfect basis of a $U_q(\mathfrak{g})$-representation [BK07, Def. 5.30]. We relate this to crystal bases by showing they give isomorphic structures as crystals, thereby avoiding the necessity of the quantum groups.

Talk 6: The crystal $B(\infty)$ (Alex)

The crystal $B(\infty)$ corresponds to the crystal basis of the lower half of the quantum group. We discuss the direct limit construction, a characterization of $B(\infty)$, and how one can recover highest weight crystals $B(\lambda)$ from $B(\infty)$ [Nak99, Thm. 3.1], analogous to highest weight irreducible representations from Verma modules. See [BS17, Ch. 12]. Alternative approach using rigged configurations [SS15].

Talk 7: Combinatorial models (Seamus)

We give the Kashiwara–Nakashima tableaux models for the crystal $B(\lambda)$ [BS17, Ch. 3,6],[HK02, Ch. 7-8] and $B(\infty)$ in the (marginally) large tableaux [Cli98, HL08, KN94]. If time permits, the Littelmann path model [Lit95a, Lit95b].

Talk 8: Littlewood–Richardson rule (Sin.)

The theory of characters is an important aspect of representation theory, and we first show that Schur functions [Sta99, §7.10] can be defined as the character of a $\mathfrak{sl}_n$ crystal. Then, using the crystal structure, we derive the Littlewood–Richardson rule using highest weight crystals, RSK insertion [BS17, §7.1][Sta99, §7.11], and jeu de taquin [Sta99, §A1.2]. See also [BS17, Ch. 9] and [Sta99, App. A].

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