GRADUATE STUDIES A NOTES

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Abstract. These are notes for Graduate Studies A, Topics In Lie Theory. Please let me know of any errors or typos you might spot. This document makes no guarantee of being comprehensive. Proofs may be terse and require details to be filled in (which are good exercises for students and non-students alike).

Contents

1. Introduction

Ostensibly the goal of this course is to explain parts of Lie theory that are not covered in the standard M.Sc course sequences. Because of the desire to get to certain places, many proofs will not be covered. We start with algebraic groups.

Exercise 1.1. *Fill in all missing/sketched proofs.*

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2. Algebraic Geometry

Some level of algebraic geometry is desired to understand algebraic groups. If you only ever work over $\mathbb{C}$, then you can always work with complex analytic Lie groups and you don't need any algebraic geometry. But if you want to work over any other field, some algebraic geometry is necessary.

Classical books on algebraic geometry by Borel, Humphreys and Springer all work with varieties. Familiarity with schemes will help at times.

Definition 2.1. *Let X be a scheme and R a ring. Define*

$$X(R) = \mathrm{Hom}(X, \mathrm{Spec}(R)).$$

The hom set above should be taken in the category of schemes (or in the category of schemes over S if that is the more natural setting).

Since the determinant is a polynomial in the entries of the matrix, we have the following example:

Example 2.2. *Let $X = \mathit{Spec}(\mathbb{Z}[\{x_{ij}\}_{i,j=1}^n)]/(\det -1))$. Then $X(R)$ is the set of $n \times n$ matrices whose determinant is equal to 1.*

3. Algebraic Groups

We can define the notion of a group object in any category which admits products and has a terminal object.

Definition 3.1. *A group object in a category $\mathcal{C}$ is an object G together with morphisms $m : G \times G \to G$, $\iota : G \to G$ and $e : * \to G$ satisfying the obvious conditions.*

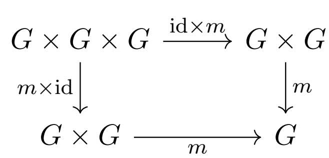

Figure 1. Associativity of multiplication

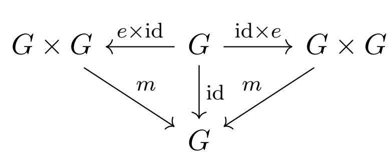

Figure 2. Unit axiom

Missing is the opposite inverse axiom. You can draw that for yourself.

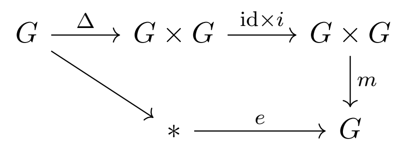

FIGURE 3. Inverse axiom

Definition 3.2. *A group scheme is a group object in the category of schemes.*

Definition 3.3. *An algebraic group is a group object in the category of algebraic varieties.*

Milne [M] chooses a different definition. The difference only manifests itself over a field of positive characteristic.

Definition 3.4. *An algebraic group is a group scheme of finite type over a field.*

Definition 3.5. *An affine algebraic group is an algebraic group that is affine.*

Example 3.6. *The group GL_n, is Spec $(k[\{x_{ij}\}_{i,j=1}^n, t]/(t\det -1))$. With this definition we have $GL_n(R)$ is the group of $n \times n$ invertible matrices with entries in R.*

Technically in the above example we should've given the maps m, ι and e. They all exist as morphisms of algebraic varieties because matrix multiplication is a polynomial in the entries of the matrix, the determinant of a product is the product of determinants, and the inverse of a matrix is a polynomial in the entries of the matrix together with the inverse of the determinant.

Definition 3.7. *A linear algebraic group is a closed subgroup of GL_n for some n.*

You will often see the adjectives linear and affine interchanged because of the following theorem.

Theorem 3.8. *An algebraic group is affine if and only if it is linear.*

There exist algebraic groups that are not affine (e.g. elliptic curves, or more generally abelian varieties). Their study is very orthogonal to the affine case, we will not talk about them further.

The other families of classical groups are associated to stabilisers of bilinear forms on vector spaces.

Example 3.9. *Let V be a finite dimensional vector space, and let $\langle \cdot, \cdot \rangle$ be a non-degenerate symplectic bilinear form on V. The symplectic group $Sp(V)$ is the stabiliser in $GL(V)$ of this bilinear form.*

Concretely, every symplectic bilinear form is of the form $\langle \mathbf{v}, \mathbf{w} \rangle = {}^t\mathbf{v}J\mathbf{w}$ for some matrix J satisfying ${}^tJ = -J$. The nondegeneracy condition is that $\det J \neq 0$. For this to happen it is necessary that V is even-dimensional.

With this, a matrix g lies in $Sp(V)$ if $\langle g\mathbf{v}, g\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in V$, which is equivalent to ${}^tgJg = J$. This last equation is a polynomial condition on the entries of g, showing that $Sp(V)$ is a closed subgroup of $GL(V)$.

The "best" choice of J is $J = \left(\begin{smallmatrix} 0 & j \\ -j & 0 \end{smallmatrix}\right)$ with j antidiagonal with 1's on the antidiagonal. All other choices of J are equivalent to this one via a change of basis. With this choice, we also write Sp_{2n} for $Sp(V)$ where $2n = \dim V$.

The other classical groups that exist over any field are the orthogonal groups. Let V be a vector space and q a quadratic form. The orthogonal group is the stabiliser of the quadratic form and the special orthogonal group is the subgroup of the orthogonal group consisting of elements with determinant 1 (we ignore characteristic two for now, if you want to see that case with all the details done properly, see [C]).

Standard choices of a quadratic form are

$$q(x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n) = x_0^2 + \sum_{i=1}^{n} x_i x_{-i}$$

in odd dimensions (type B) and

$$q(x_{-n}, \dots, x_{-1}, x_1, \dots, x_n) = \sum_{i=1}^{n} x_i x_{-i}$$

in even dimensions (type D).

4. Semisimple and Unipotent elements

First we define semisimple and unipotent.

Definition 4.1. *An element $g \in GL_n(k)$ is semisimple if it is diagonalisable after passing to the algebraic closure. An element $g \in GL_n(k)$ is unipotent if its only eigenvalue (after passing to an algebraic closure) is 1.*

Theorem 4.2. *Let $g \in GL_n(k)$. Then there exist unique $s, u \in GL_n(k)$ such that $g = su$, s is semisimple, u is unipotent and s and u commute.*

The following is not quite a proof, but it works in characteristic zero. Exercise: work out what goes wrong.

Proof. Uniqueness is clear as the only element that is semisimple and unipotent is the identity. Over an algebraically closed field, existence follows from Jordan normal form. Now if $g \in GL_n(k)$, by the algebraically closed field case, we can write $g = su$ with $s, u \in GL_n(\overline{k})$. By uniqueness, s and u are invariant under the Galois group so lie in $GL_n(k)$. □

5. Group actions

The definition of an algebraic group G acting on a variety X consists of the datum of an action map $a : G \times X \to X$ such that the obvious diagrams commute.

Exercise 5.1. *If G and X are affine, express the datum of an action of G on X purely in terms of the algebras $k[G]$ and $k[X]$ (you should discover the notion of a comodule).*

Exercise 5.2. *Let X be affine. Show that the datum of a $\mathbb{G}_m$-action on X is equivalent to a $\mathbb{Z}$-grading on $k[X]$.*

6. Borel subgroups

Work over an algebraically closed field.

Definition 6.1. *Let G be a linear algebraic group. A Borel subgroup is a maximal solvable connected subgroup of G.*

Here are some theorems about Borel subgroups.

Theorem 6.2. *Every two Borel subgroups are conjugate by an element of $G(k)$.*

Theorem 6.3. *Assume G is connected. A Borel subgroup is its own normaliser.*

Theorem 6.4. *A connected solvable affine group acting on a proper variety has a fixed point.*

Proof. Let G be the group and X the variety. First we deal with the case where the dimension of G is 1. Let $x \in X$. Let $\bar{G}$ be the completion of G (as a variety, so it'll always be $\mathbb{P}^1$). There is $\varphi_x : G \to X$ given by $\varphi_x(g) = g \cdot x$. As X is proper we can extend φ_x to a morphism from $\bar{G}$ to X. The image of any point in $\bar{G} \setminus G$ is a fixed point.

If N is a connected normal subgroup of G then $X^G = (X^N)^{G/N}$. So we can proceed by induction on the dimension of G if G contains a connected normal proper subgroup of positive dimension. This is always the case (we omit this for now). □

As a corollary, we obtain the Lie-Kolchin theorem. (for statements about flag varieties, look to §12).

Theorem 6.5. *Every smooth connected solvable subgroup of $GL(V)$ stabilises a complete flag.*

This theorem is equivalent to saying that such a subgroup is conjugate to a subgroup of upper-triangular matrices.

For examples showing the necessity of the various assumptions in the Lie-Kolchin theorem, see [M, 16.36-16.38].

Examples of Borel subgroups:

If $G = GL_n$, then a Borel subgroup is given by the subgroup of upper-triangular matrices. (note a Borel can't be any bigger by the Lie-Kolchin theorem, giving the proof).

If G is one of the other classical groups given in their standard realisations as defined earlier, then a Borel subgroup is again given by the subgroup of upper-triangular matrices. A word of warning, the truth of this fact depends on the choice of the matrix J (in the symplectic case) or the quadratic form q (in the orthogonal case).

Exercise 6.6. *In Sp_{2n} consider the elements*

$$e_i(x) = I + x(E_{i,i+1} - E_{2n-i,2n-i+1})$$

for $1 \leq i < n$ and

$$e_n(x) = 1 + xE_{n,n+1},$$

where $x \in k$.

1. *Show that $e_i(x) \in Sp_{2n}$.*
2. *Show that the only complete flag stabilised by all $e_i(x)$ with $1 \leq i \leq n$ and $x \in k$ is the usual one.*
3. *Conclude that the subgroup of upper-triangular matrices in Sp_{2n} is a Borel subgroup.*

7. Reductive groups

Definition 7.1. *Let G be a linear algebraic group. Its unipotent radical is the maximal connected normal unipotent subgroup of G.*

The unipotent radical always exists.

Definition 7.2. *A group G is reductive if its unipotent radical is trivial.*

Remark 7.3. Some authors may assume that a reducitve group is also connected. This is the most important case.

8. roots

Let G be a reductive group. Let T be a maximal torus of G. Let $\mathfrak{g} = \mathrm{Lie}(G)$. Consider the adjoint action of T on $\mathfrak{g}$. Then there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}(t)(x) = \alpha(t)x \text{ for all } t \in T\} \neq 0.$$

This defines a subset $\Phi \subset X^*(T)$, called the set of roots.

It is a general fact about torus actions that such a decomposition exists, what is less trivial is that every $\mathfrak{g}_\alpha$ is one-dimensional.

9. Facts about the Weyl group

For distinct $\alpha, \beta \in \Delta$, let $m_{\alpha\beta}$ be the order of $s_\alpha s_\beta$. Then we have

Theorem 9.1. *The Weyl group W has a presentation*

$$W = \langle \{s_\alpha\}_{\alpha \in \Delta} \mid s_\alpha^2 = 1, (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1 \rangle,$$

i.e. W is a Coxeter group.

Definition 9.2. *Let $w \in W$. Define the length of w, denoted $\ell(w)$, to be the smallest number ℓ such that we can write $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell}$ with $\alpha_i \in \Delta$ for all i.*

In the above, we say that $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell}$ is a reduced expression for w. Reduced expressions are not unique, but you can always get from one to another using only the braid relations (Matsumoto's theorem).

For $w \in W$, define $\Phi_w = \{\alpha \in \Phi^+ \mid w\alpha \in \Phi^-\}$. The following description of Φ_w can be useful (and is a good exercise to try to prove).

Theorem 9.3. [B, Ch VI, §6, Corollarie 2] *Let $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of w. Then*

$$\Phi_w = \{\alpha_{i_k}, s_{i_k}\alpha_{i_{k-1}}, s_{i_k}s_{i_{k-1}}\alpha_{i_{k-2}}, \ldots, s_{i_k} \cdots s_{i_2}\alpha_{i_1}\}.$$

In particular $|\Phi_w| = \ell(w)$.

10. Unipotent groups

Theorem 10.1. *Let X be a set of roots closed under addition, and such that if $\alpha \in X$ then $-\alpha \notin X$. Then there is a unipotent subgroup U_X of G whose Lie algebra is the sum of the root subspaces of G contained in X. Furthermore for any choice of order on the product, the multiplication map*

$$\prod_{\alpha \in X} U_\alpha \to U_X$$

is an isomorphism.

Theorem 10.2. *For all roots α and β with $\alpha + \beta \neq 0$, there is an identity*

$$e_\alpha(x)e_\beta(y) = \Big[\prod_{\substack{i,j \in \mathbb{Z}^+ \\ i\alpha + j\beta = \gamma \in \Phi}} e_\gamma(c_{i,j,\alpha,\beta} x^i y^j) \Big] e_\beta(y) e_\alpha(x),$$

for some constants $c_{i,j,\alpha,\beta}$.

Remark 10.3. Note that the product in the above theorem is a product in a non-commutative group. One needs to first choose an order of factors in the product, then the result is true. The constants $c_{i,j,\alpha,\beta}$ can depend on this order chosen.

Exercise 10.4. *Find all root systems where the elements in this product are actually non-commutative.*

Remark 10.5. With a good choice of scaling of e_α, it can be ensured that $c_{i,j,\alpha,\beta} \in \{-3,-2,-1,1,2,3\}$. This requires more work.

Proof. (Sketch) Find λ such that $\langle \alpha, \lambda \rangle$, and $\langle \beta, \lambda \rangle$ are both positive so we are sure we are in a unipotent group. Perform an induction on dimension to get a formlua of the form

$$e_\alpha(x)e_\beta(y) = \Big[\prod_{\substack{i,j \in \mathbb{Z}^+ \\ i\alpha + j\beta = \gamma \in \Phi}} e_\gamma(f_{i,j,\alpha,\beta}(x,y)) \Big] e_\beta(y) e_\alpha(x),$$

and study the torus action on this identity to show the $f_{i,j,\alpha,\beta}(x,y)$ are monomials. □

11. representation theory

Definition 11.1. *A representation of G is a vector space V, together with a morphism $a : G \times V \to V$ which satisfies the expected axioms.*

The usual definitions of submodule and simple/irreducible module apply.

We will always consider finite dimensional representations.

Definition 11.2. *A category is semisimple if every object is a direct sum of simple objects.*

Theorem 11.3. *If k has characteristic zero the category of representations of G is semisimple if and only if G is reductive.*

We'll only talk about the direction G reductive implies the category is semisimple. One way to prove this is to bootstrap the corresponding semisimplicity result for Lie algebras: Let W be a G-invariant subspace of V. Then from the Lie algebra theory, there exists a complementary $Lie(G)$-invariant subspace. This is then invariant under G^0, which proves the theorem for connected groups. To pass from connected groups to disconnected groups, one can mirror one of the arguments from finite group theory (the one where you pick a complementary vector space and then average it). The above discussed approach is what you see in Milne. Another approach goes via Weyl's unitary trick (WLOG $k = \mathbb{C}$ by Lefschetz principle), bootstrapping off the corresponding result for compact groups.

We won't use it, but for completeness here is the characteristic p result:

Theorem 11.4. *If k has characteristic $p > 0$ the category of representations of G is semisimple if and only if G^0 is a torus and $|G/G^0|$ has order prime to p.*

12. The flag variety

Let G be a connected linear algebraic group, and let B be a borel subgroup.

Definition 12.1. *The flag variety of G is the variety G/B.*

Definition 12.2. *The flag variety of G is the variety parametrising Borel subgroups of G.*

The equivalence of these two definitions follows from the two fundamental facts about conjugacy of Borel subgroups.

Remark 12.3. If H is any (closed) subgroup of G, we can turn G/H into a quasi-projective variety.

Remark 12.4. Technically, the second definition of flag variety is not a definition, since it doesn't explain the structure of an algebraic variety.

Definition 12.5. *A flag in a vector space V is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$. If $\dim V_i = i$ for all i, then the flag is called a complete flag.*

Example 12.6. *If $G = GL_n$, then the flag variety is the variety Fl_n of all complete flags in k^n.*

In this case, there is a classical method to realise Fl_n as a projective variety, considering the composite of the embeddings

$$Fl_n \hookrightarrow \prod_{i=1}^{n-1} \mathrm{Gr}(i,n) \hookrightarrow \prod_{i=1}^{n-1} \mathbb{P}(\wedge^i k^n) \hookrightarrow \mathbb{P}^N.$$

Here $\mathrm{Gr}(i,n)$ is the Grassmannian variety parametrising i-dimensional subspaces of k^n and the first embedding is the obvious one. The second embedding is the product of the Plucker embeddings of the Grassmannian, it sends a subspace W to the line $\wedge^i W$. The final embedding is the Segre embedding. Something needs to be done at each stage to prove that these are closed embeddings, this is all standard classical algebraic geometry.

13. TORUS ACTIONS ON PROJECTIVE VARIETIES

References [BB] and [CG, §2.4]

14. PICARD GROUP OF THE FLAG VARIETY

The Picard group of a variety is the group of line bundles on it.

There is a natural map

$$X^*(H) \to \operatorname{Pic}(G/B).$$

One way to think of this: The group H is canonically a quotient of each Borel subgroup. So given $\lambda : H \to \mathbb{G}_m$, we pullback to obtain a one-dimensional representation of that Borel. This is a continuously varying family of representations so defines a line bundle over the variety of Borel subgroups.

Here is another way: Consider the quotient of $G \times \mathbb{A}^1$ by the B action given by $(g, x) \sim (gb, \lambda(b)x)$. This is a line bundle over G/B.

This map from $X^*(H)$ to $\operatorname{Pic}(G/B)$ is not an isomorphism (though it is an isomorphism if you look at equivariant line bundles, and you need equivariant line bundles if you want Borel-Weil-Bott). It is an isomorphism if G is semisimple and simply-connected.

15. BOREL-WEIL-BOTT

References: [L] and [J, §5]

Exercise 15.1. *Give an explicit formula for the highest weight vector as a function on G as a product of minors when $G = GL_n$. Generalise this to a statement about the highest weight vector function being a product of generalised minors for fundamental weights for arbitrary reductive G.*

REFERENCES

[BB] Bialynicki-Birula - Some theorems on actions of algebraic groups. 9
[B] Bourbaki - Groupes et Algebres de Lie (book). 6
[C] Conrad - Reductive Group Schemes. 4
[CG] Chriss and Ginzburg - Representation Theory and Complex Geometry - book. 9
[J] Jantzen - Representations of Algebraic Groups - book. 9
[L] Lurie - A proof of the Borel-Weil-Bott theorem 9
[M] Milne - Algebraic Groups (book). 3, 5

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