Homework 3

Due: Thursday April 25, 2013

1. Find all zeros and poles of the function

$$\cot(z) = i\frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}$$

   in the complex plane and determine their order.

2. Let f be a meromorphic function with at most a simple pole at $z_0 \in \mathbb{C}$. Fix $0 \leq \theta \leq 2\pi$. Let C_ϵ be an arc of a circle of radius ϵ, centred at z_0, of angular width θ. Prove that

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} f(z)dz = i\theta \operatorname{Res}_{z_0} f.$$

3. Evaluate the integral

$$\int_0^\infty \frac{1}{1 + x^4} dx$$

4. Evaluate the integral

$$\int_0^\infty \frac{\sin(x)}{x} dx.$$

   This integral is not absolutely convergent, but we can interpret it as the limit as R tends to infinity of the integral over the segment $[0, R]$.

   For a hint, consider a rectangular contour with one side along the x-axis and a small semicircle cut out at the origin. The function e^{iz}/z has the integrand as its imaginary part when z is real.