1(a): The function $\frac{z^2}{(z+2)^3}$ is holomorphic and zero at $z=0$, so $z^{-2}(z+2)^3$ has a pole at $z=0$.

$$z^{-2}(z+3)^3 = \frac{8}{z^2} + \frac{12}{z} + 6 + z.$$

The residue is equal to the coefficient of z^{-1} in the Laurent expansion, hence is 12.

(b) $\Gamma(z) = \frac{\Gamma(z+1)}{z}$. $\Gamma(z+1)$ is holomorphic at $z=0$, and $\Gamma(0+1)=\Gamma(1)=1$, thus $\Gamma(z)$ has a simple pole at $z=0$. The residue is equal to

$$\lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0}\Gamma(z+1) = \Gamma(1) = 1.$$

(c). If $z\in\mathbb{R}$, $z\neq 0$, then $\left|\sin\left(\frac{1}{z}\right)\right| \leq 1$. Therefore it cannot be true that $\lim_{z\to 0}\left|\sin\left(\frac{1}{z}\right)\right| = \infty$. Thus $\sin\left(\frac{1}{z}\right)$ does not have a pole at $z=0$

(d) It was shown in class that $\xi(z) = \pi^{-z}\,\Gamma\left(\frac{z}{2}\right)\zeta(z)$ has a simple pole at $z=0$. The function π^{-z} is nonzero and holomorphic while the function $\Gamma\left(\frac{z}{2}\right)$ has a simple pole at $z=0$. Therefore $\zeta(z)$ cannot have a pole at $z=0$.

Q2: Without loss of generality assume $z_0=0$.
In some ball centred at the origin, $f(z)$ is expressible as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Let k be the smallest index for which $a_k\neq 0$.
Let $g(z) = \sum_{n=k}^{\infty} a_n z^{n-k}$. Then $f(z) = z^k g(z)$ and $g(0)=a_k\neq 0$.

The power series for $g(z)$ converges absolutely wherever that for f does, hence $g(z)$ is an analytic function on some disc centred at the origin.
As g is continuous and $g(0)\neq 0$, there exists an open set $U\ni 0$ such that $g(z)\neq 0$ for all $z\in U$. (e.g. $U = g^{-1}(\mathbb{C}\setminus\{0\})$)
Then if $z\in U$, $f(z)=z^k g(z)$ is zero if and only if $z=0$. So for a disc centred at 0 contained in U (which exists as U is open), the only zero of f in this disc is at 0.

3): Let $f_k(z) = \sum_{\substack{n \in \mathbb{Z} \\ |n| < k}} \cancel{} \frac{1}{(z+n)^2}$. We will show that

f_k converges to f uniformly on compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

Let K be a compact subset of $\mathbb{C}$. Then K is bounded so there exists $R \in \mathbb{R}$ such that $|z| \le R$ for all $z \in K$.

If $|n| > R$ and $|z| \le R$ we have

$$\left| \frac{1}{(z+n)^2} \right| \le \frac{1}{(|n|-R)^2}.$$

Now if $l > k > R$, we have, for all $z \in K$:

$$\left| f_l(z) - f_k(z) \right| \le \sum_{n=k+1}^{l} \frac{2}{(n-R)^2} \quad (*).$$

~~We~~ Since $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges, this estimate implies that $f_k(z)$ converges to $\sum_{n \in \mathbb{Z}} \frac{1}{(n+z)^2}$ (which itself was just shown to be absolutely convergent).

Since the bound in $(*)$ is independent of $z \in K$, this convergence is uniform on K.

Since a limit of analytic functions which converges uniformly on compact sets is itself analytic, we have shown that the function $f(z)$ is analytic on $\mathbb{C} \setminus \mathbb{Z}$.

Now consider $f(z)$ for $|z-n| < 1/2$ for some integer n.

Write $$f(z) = \frac{1}{(z-n)^2} + \sum_{\substack{m \in \mathbb{Z} \\ m \neq -n}} \frac{1}{(z+m)^2}.$$

The same argument shows that the function $g := \sum_{\substack{m \in \mathbb{Z} \\ m \neq -n}} \frac{1}{(z+m)^2}$ is analytic in a neighbourhood of n.

The function $\frac{1}{(z-n)^2}$ is meromorphic, and hence $f(z)$ is meromorphic.

4: Write $\dfrac{e^{-z-z^2/2}}{1-z} = \dfrac{e^{-3/2}}{1-z} + \dfrac{e^{-z-z^2/2} - e^{-3/2}}{1-z}$.

Let $g(z) = \sum_{n=0}^{\infty} b_n z^n = \dfrac{e^{-z-z^2/2} - e^{-3/2}}{1-z}$.

At $z=1$, the numerator is zero, so the potential pole at $z=1$ is cancelled by the numerator. Hence $g(z)$, which was decoly meromorphic is actually holomorphic on all of $\mathbb{C}$

$$\therefore \lim_{n\to\infty} b_n = 0.$$

The power series expansion for $\dfrac{e^{-3/2}}{1-z}$ is $\sum_{n=0}^{\infty} e^{-3/2} z^n$.

$$\therefore a_n = e^{-3/2} + b_n.$$

$$\therefore \lim_{n\to\infty} a_n = e^{-3/2}$$

$\therefore$ An asymptotic formula for a_n is $e^{-3/2}$.