Solutions to 116 Homework 1

1. Write $z = x + iy$.

(a). If $z_1 = z_2$, all of $\mathbb{C}$. Otherwise, the perpendicular bisector of the line segment connecting z_1 and z_2.

(b). The unit circle.

(c). The vertical line passing through 3.

(d). The closed half-plane above the line $y = c$.

(e). If $a = a_1 + ia_2$ and $b = b_1 + ib_2$, then the the open half-plane defined by

$$a_1 x - a_2 y + b_1 > 0$$

(f). The parabola $y^2 = 2x + 1$.

2. Suppose $r = 0$. If $n \leq 0$ there is no solution (I treat 0^0 as undefined), but if $n > 0$ the solution set is $\{0\}$.

Take $r > 0$. If $n = 0$ then there is a solution precisely when $w = 1$, in which case the solution set is $\mathbb{C} - \{0\}$. Otherwise, there are $|n|$ solutions, given by

$$z = \rho e^{i(\phi + 2\pi k)/n} \quad k \in \{0, \ldots, |n| - 1\}$$

where $\rho = r^{1/n}$ is positive, real n-th root.

3. The domain $\Omega = \{re^{i\theta} : r > 0, \theta \in (-\pi, \pi)\}$ is diffeomorphic to $(0, \infty) \times (-\pi, \pi)$ via the transformation

$$g : (0, \infty) \times (\pi, \pi) \to \Omega : (r, \theta) \mapsto (r \cos\theta, r \sin\theta)$$

In fact we only need that g is a local diffeomorphism. This follows because the Jacbian

$$Dg = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

is invertible for every $(r, \theta) \in (0, \infty) \times (-\pi, \pi)$. By the inverse function theorem, g^{-1} is locally a diffeomorphism, with Jacobian

$$Dg^{-1} = (Dg)^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}$$

Write $u = \log r$ and $v = \theta$, so that $\log z = u + iv$. Then the above shows u, v are C^1 functions of (x, y). We calculate the derivatives of u, v.

$$Du(x, y) = Du(r, \theta) Dg^{-1}(x, y) = \begin{pmatrix} \frac{1}{r}\cos\theta & \frac{1}{r}\sin\theta \end{pmatrix}$$

$$Dv(x, y) = Dv(r, \theta) Dg^{-1}(x, y) = \begin{pmatrix} -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}$$

Therefore $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. So u, v are C^1 and satisfy the Cauchy-Riemann equations. Therefore $\log z$ as defined is holomorphic on Σ.

4. Suppose $f(z) = \sum_n a_n z^n$ has radius of convergence $R > 0$. Pick z_0 in this radius. Then

$$f(z) = f(z - z_0 + z_0) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n \binom{n}{k} (z - z_0)^k z_0^{n-k} \tag{1}$$

If $k > n$ we define $\binom{n}{k} = 0$. Note we can pull the a_n inside because the inner-most sum is actually finite.

Take $|z - z_0| < \rho < R - |z_0|$. Then

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left| a_n \binom{n}{k} (z - z_0)^k z_0^{n-k} \right| = \sum_{n=0}^{\infty} |a_n| (|z - z_0| + |z_0|)^n \leq \sum_{n=0}^{\infty} |a_n| \rho^n < \infty$$

So the double sum of (1) converges absolutely. In particular, the real and imaginary parts converge absolutely. By Fubini/Tonelli we are justified in changing the order of summation. Therefore

$$f(z) = \sum_{k=0}^{\infty} (z - z_0)^k \sum_{n=0}^{\infty} a_n \binom{n}{k} z_0^{n-k} = \sum_{k=0}^{\infty} b_k (z - z_0)^k$$

where $b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k}$.

So about each z_0 in the radius of convergence f has a power series centered at z_0 with radius of convergence $R - |z_0|$.

5. Let $P(z) = a_n z^n + \ldots + a_0$. Write

$$P(z) = a_n z^n (1 + \frac{a_{n-1}}{a_n} z^{-1} + \ldots + \frac{a_0}{a_n} z^{-n})$$

Then for $|z|$ sufficiently large, $|P(z)| \geq |a_n||z|^n (1 - \frac{1}{2})$. This shows that the infimum of $|P|$ is the infimum of $|P|$ restricted to some ball of radius R. Now $\overline{B_R(0)}$ is compact, and $|P|$ is continuous, so the the infimum is reached at some $z_0 \in B_R(0)$.

Suppose, towards a contradiction, that $P(z_0) \neq 0$. Expand $P(z)$ as a Taylor series about z_0:

$$P(z) = \sum_{k=0}^{n} c_k (z - z_0)^k$$

Let k be the least non-zero integer so that $c_k \neq 0$. P is assumed non-constant, so k exists. Let α be a k-th root of $-\frac{c_0}{c_k}$, and take $M = |c_{k+1}| + \ldots + |c_n|$. Then for $\epsilon \ll 1$

$$\begin{aligned} |P(z_0 + \epsilon\alpha)| &\leq |c_0 + c_k \epsilon^k \alpha^k| + |\epsilon\alpha|^{k+1} M \\ &= |c_0|(1 - \epsilon^k) + \epsilon^{k+1} M' \end{aligned}$$

Choose $\epsilon > 0$ so that $\epsilon M' < \frac{1}{2}$, then we have

$$|P(z_0 + \epsilon\alpha)| \leq 1 - \tfrac{1}{2}\epsilon^k < |P(z_0)|$$

This is a contradiction. So $P(z_0)$ must be 0.