Solutions to 116 Homework 2

1. Write $z(t) = x(t) + iy(t)$, and $F(x+iy) = U(x,y) + iV(x,y)$. Since F is holomorphic in $\Omega \ni z(c)$, its component functions U, V are differentiable (in fact analytic) in a neighborhood of $z(c)$. By assumption, z is differentiable at c.

So $F \circ z : [a,b] \to \mathbb{C}$ is differentiable at c, with derivative given by the chain rule as

$$(F \circ z)' = U_x x' + U_y y' + i(V_x x' + V_y y')$$

where we write U_x for $\frac{\partial U}{\partial x}(z(c))$, etc.

Using the Cauchy-Riemann equations, we can rewrite the above as

$$\begin{aligned}(F \circ z)'(c) &= U_x x' - V_x y' + i(V_x x' + U_x y') \\ &= F'(z(c))z'(c)\end{aligned}$$

2. Recall the ratio test: if $f = \sum_{n=0}^{\infty} a_n z^n$, and the limit

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists (possibly infinite), then the radius of convergence of f is $1/L$. Here we define the ratios $1/\infty = 0$ and $1/0 = \infty$.

In each problem we consider $|\frac{a_{n+1}}{a_n}|$ as n tends to ∞. Write R for the radius of convergence.

(a). Using that log is bounded near 1,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{\log(n+1)}{\log n} \right)^2 = \left(1 + \frac{\log \frac{n+1}{n}}{\log n} \right)^2 \longrightarrow 1$$

So $R = 1$.

(b).

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{n!} = n \longrightarrow \infty$$

So $R = 0$.

(c).

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{3^{n+1} + 2(n+1)} \frac{3^n + 2n}{n^2} = 1/3 \frac{1 + 3^{-n}2n}{1 + 3^{-n}2(n+1)} \frac{(n+1)^2}{n^2} \longrightarrow 1/3$$

So $R = 3$.

(d).

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{(3n+3)!} \frac{(3n)!}{(n!)^3} = \frac{(n+1)^3}{(3n+1)(3n+2)(3n+3)} \longrightarrow 1/27$$

So $R = 27$.

3. Trivially $f(z) = z^2$ is entire. So given any w and any circle C containing w, the Cauchy integral formula says

$$f(w) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - w} dz$$

where C is oriented counter-clockwise. In this problem $w = 1$ and $C = \{|z| = 3\}$. So

$$\int_{|z|=3} \frac{z^2}{z-1} dz = 2\pi i * 1^2 = 2\pi i$$

4. Since f is entire, it has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that converges on all of $\mathbb{C}$. And

$$a_n = \frac{f^{(n)}(0)}{n!}$$

Let C_R be the circle of radius. By the Cauchy integral formula,

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{z^{n+1}} dz$$

Let $P(x) = c_0 + \ldots + c_d x^d$ be a real polynomial of degree d. Set $M = |c_0| + \ldots + |c_d|$. Then for $R \gg 1$,

$$\begin{aligned} |a_n| &\leq \frac{1}{2\pi} \int_{C_R} \frac{|f(z)|}{|z|^{n+1}} dz \\ &\leq \frac{1}{2\pi R^{n+1}} \int_{C_R} P(R) dz \\ &\leq M R^{d-n} \end{aligned}$$

So if $n > d$, taking R arbitrarily large shows that $a_n = 0$. Therefore $f(z) = a_0 + \ldots + a_d z^d$ is a polynomial of at most degree d.