Inequalities (19 May 2018)
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Some Theory

This is not an exhaustive list by any stretch of the imagination.

The most basic inequality is

$$x^2 \geqslant 0$$

Always keep in mind what you know and what you have to prove. It is very frustrating to spend lots of time proving $a \geqslant c$ and $b \geqslant c$ when the question wants you to prove $a \geqslant b$, since you have made zero progress.

The first non-trivial inequality usually encountered is the AM-GM inequality:

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \cdots x_n}$$

for all non-negative reals $x_1, x_2, \ldots, x_n$, with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Various generalisations to look up: Quadratic mean $\geqslant$ Arithmetic mean $\geqslant$ Geometric mean $\geqslant$ Harmonic mean. This sequence of inequalities generalises to the Power mean inequality. Muirhead's inequality is also a generalisation, though is often simply provable by repeated applications of AM-GM.

Jensen's Inequality:

If f is a convex function on an interval I, then

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \leqslant \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

for all $x_1, x_2, \ldots, x_n \in I$. To apply this, you need to know what a convex function is. The following are equivalent (for reasonable functions)

- $f(x)$ is convex
- $f(\frac{x+y}{2}) \leqslant \frac{f(x)+f(y)}{2}$ for all $x, y \in I$.
- $f''(x) \geqslant 0$ for all $x \in I$.

There is a corresponding version for concave functions with all the inequality signs reversed.

Geometric Inequalities:

If a, b and c are the sides of a triangle, always make the substitution $a = y + z$, $b = z + x$, $c = x + y$. Then by the triangle inequality, x, y, and z are all positve real numbers.

Problems

(1) If $x + y = 1$ and $x > 0$, $y > 0$, prove that
$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) \geqslant 9.$$

(2) Prove the following inequality for all positive real numbers a, b and c:
$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right) \geqslant 8$$

(3) If a, b and c are all positive reals, prove that
$$a^7 + b^7 + c^7 \geqslant a^4b^3 + b^4c^3 + c^4a^3.$$

(4) Find all real solutions to the following system of equations
$$x + y = 2$$
$$xy - z^2 = 1.$$

(5) Use Jensen's inequality for $f(x) = \log(x)$ to prove the AM-GM inequality.

(6) If a, b and c are the lengths of the sides of a scalene triangle, prove that
$$\frac{1}{b+c-a} + \frac{1}{c+a-b} + \frac{1}{a+b-c} > \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(7) (Nesbitt) Let a, b and c be positive real numbers. Prove that
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geqslant \frac{3}{2}.$$

(8) Prove that $n! \geqslant n^{n/2}$.

(9) Prove that $1 \cdot 3 \cdot 5 \cdots (2n-1) \leqslant n^n$.

(10) Amongst all triangles with the same perimeter, show that the equilateral triangle is the one with the greatest area.

(11) Find the least value of $3x + 4y$, if x and y are positive numbers satisfying $x^2y^3 = 6$.

(12) (2000 AMO Q3) Let $x_1, x_2, \ldots x_n$ and $y_1, y_2 \ldots, y_n$ be real numbers such that
(a) $0 < x_1y_1 < x_2y_2 < \cdots < x_ny_n$ and
(b) $x_1 + x_2 + \cdots + x_i \geqslant y_1 + y_2 + \cdots y_i$ for $1 \leqslant i \leqslant n$.
Prove that
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \leqslant \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}$$
When does equality occur?

(13) (2001 IMO Q2) Prove that for all positive real numbers a, b and c:
$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geqslant 1.$$