Plane Geometry (2) (S)

1. Let $ABCD$ be a rhombus and P be a point on its side BC. The circle passing through A, B, and P intersects BD once more at the point Q and the circle passing through C, P, and Q intersects BD once more at the point R. Prove that A, R, and P lie on the one straight line. [Tournament of the Towns 1990]

2. The chord MN on the circle is fixed. For every diameter AB of the circle consider the intersection point C of the lines AM and BN and construct the line ℓ passing through C perpendicularly to AB. Prove that all the lines ℓ pass through a fixed point. [Tournament of the Towns 1991]

3. The points P and Q lie on a semi-circle with diameter UV. UP and VQ intersect at the point S, while the tangents to the semi-circle at P and Q intersect at R. Prove that $RS \perp UV$.

4. The incentre of the triangle $\triangle ABC$ is K. The midpoint of AB is C_1 and that of AC is B_1. The lines C_1K and AC meet at B_2, the lines B_1K and AB meet at C_2. If the areas of the triangles $\triangle AB_2C_2$ and $\triangle ABC$ are equal, what is the measure of $\angle CAB$? [1990 IMO Shortlist - HUN 3]

5. Let OA and OB be perpendicular rays in the circle $\mathcal{C}$ (with centre O). Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are internally tangent to $\mathcal{C}$ in A and B, and a third circle $\mathcal{C}_3$ is tangent externally to $\mathcal{C}_1$ and $\mathcal{C}_2$ in S and T, and internally in M to $\mathcal{C}$. Find the measure of $\angle SMT$. [1996 Romanian Mathematical Olympiad]

6. If $ABCDEF$ is a convex hexagon with $AB = BC$, $CD = DE$, $EF = FA$, prove that the altitudes (produced) of $\triangle BCD$, $\triangle DEF$ and $\triangle FAB$, emanating from vertices C, E, A, concur. [Polish and Austrian Olympiads 1981–1995]

7. Let ABC be an acute triangle with altitudes BD and CE. Points F and G are the feet of the perpendiculars BF and CG to line DE. Prove that $EF = DG$. [Polish and Austrian Olympiads 1981–1995]

8. Points D, E, F are chosen on the sides AB, BC, AC of a triangle ABC, so that $DE = BE$ and $FE = CE$. Prove that the centre of the circle circumscribed around triangle ADF lies on the bisector of $\angle DEF$. [USSR Olympiad 1989]

9. Two common tangents of two intersecting circles meet at a point A. Let B be a point of intersection of the two circles, and C and D be the points at which one of the tangents touches the circles. Prove that the line AB is tangent to the circle passing through B, C, and D. [USSR Olympiad 1990]

10. On the side AB of a convex quadrilateral $ABCD$ a point E, different from the vertices, is chosen. The segments AC and DE intersect at a point F. Prove that the circles circumscribed about $\triangle ABC$, $\triangle CDF$, and $\triangle BDE$ have a common point. [USSR Olympiad 1990]

11. Let triangle ABC have orthocentre H, and let P be a point on its circumcircle. Let E be the foot of the altitude BH, let $PAQB$ and $PARC$ be parallelograms, and let AQ meet HR in X. Prove that EX is parallel to AP.

12. (IMO 1996 Q2) Let P be a point inside $\triangle ABC$ such that $\angle APB - \angle C = \angle APC - \angle B$. Let D, E be the incentres of $\triangle APB$, $\triangle APC$ respectively. Show that AP, BD and CE meet in a point.

13. Let ABC be an acute-angled triangle with $BC > CA$. Let O be its circumcentre, H its orthocentre, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$.

14. In an acute triangle ABC, $AC > BC$, M is the midpoint of AB. Let AP be the altitude from A, BQ be the altitude from B, AP and BQ meet in H, and let the lines AB and PQ meet at R. Prove that the two lines RH and CM are perpendicular.

15. (IMO 1990 Q1) Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E and M intersects the lines BC and AC at F and G, respectively. If $\frac{AM}{AB} = t$, find $\frac{EG}{EF}$ in terms of t.

16. (Czech-Slovak 1999) An acute angled triangle ABC is given with altitudes AD, BE, CF. Suppose that the lines BC and EF have a point P in common and that the line through D parallel to EF intersects the line AC at a point Q and the line AB at a point R. Prove that the circumcircle of the triangle PQR passes through the midpoint of the side BC.

17. (IMO 1999 Q5) The circles Γ_1 and Γ_2 lie inside circle Γ, and are tangent to it at M and N respectively. It is given that Γ_1 passes through the centre of Γ_2. The common chord of Γ_1 and Γ_2, when extended, meets Γ at A and B. The lines MA and MB meet Γ_1 again at C and D. Prove that the line CD is tangent to Γ_2.

18. (Czech-Slovak 1999) Find all positive numbers k for which the following assertion holds: Among all triangles ABC with $|AB| = 5$ and $|AC| : |BC| = k$, the one with the largest area is the isosceles one.

19. Consider an acute angled triangle ABC such that $AC > BC$, and let M be the midpoint of AB and let CD, AP and BQ be the altitudes of the triangle. Let R be the intersection point of AB and PQ. Prove that MP is tangent to the circumcircle of DRP.