HOMOLOGICAL PROPERTIES OF FINITE TYPE KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

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Abstract. We give an algebraic construction of *standard modules*—infinite dimensional modules categorifying the PBW basis of the underlying quantized enveloping algebra—for Khovanov-Lauda-Rouquier algebras in all finite types. This allows us to prove in an elementary way that these algebras satisfy the homological properties of an "affine quasi-hereditary algebra." In simply-laced types these properties were established originally by Kato via a geometric approach. We also construct some Koszul-like projective resolutions of standard modules corresponding to multiplicity-free positive roots.

1. Introduction

Working over $\mathbb{Q}(q)$ for an indeterminate q, let $\mathbf{f}$ be the quantized enveloping algebra associated to a maximal nilpotent subalgebra of a finite dimensional complex semisimple Lie algebra $\mathfrak{g}$. It is naturally Q^+-graded

$$\mathbf{f} = \bigoplus_{\alpha \in Q^+} \mathbf{f}_\alpha,$$

where Q^+ denotes $\mathbb{N}$-linear combinations of the simple roots $\{\alpha_i \mid i \in I\}$. Moreover $\mathbf{f}$ is equipped with several distinguished bases, including Lusztig's *canonical basis* (Kashiwara's lower global crystal base) and various *PBW bases*, one for each choice $\prec$ of convex ordering of the set R^+ of positive roots. Passing to dual bases with respect to Lusztig's form $(\cdot,\cdot)$ on $\mathbf{f}$, we obtain the *dual canonical basis* (Kashiwara's upper global crystal base) and some *dual PBW bases*. See [L1], [L2] and [K]. Lusztig's approach gives a *categorification* of $\mathbf{f}$ in terms of certain categories of sheaves on a quiver variety. Multiplication on $\mathbf{f}$ comes from Lusztig's induction functor and (at least in simply-laced types) the canonical basis arises from the irreducible perverse sheaves in these categories.

In 2008 Khovanov and Lauda [KL1, KL2] and Rouquier [R1] introduced for any field $\mathbb{K}$ a (locally unital) graded $\mathbb{K}$-algebra

$$H = \bigoplus_{\alpha \in Q^+} H_\alpha,$$

known as the *Khovanov-Lauda-Rouquier algebra* (KLR for short). Let $\mathrm{Proj}(H)$ be the additive category of finitely generated graded projective left H-modules. We make the split Grothendieck group $[\mathrm{Proj}(H)]$ of this category into a $\mathbb{Z}[q,q^{-1}]$-algebra, with multiplication arising from the induction product $\circ$ on modules over the KLR algebra

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and action of q induced by upwards degree shift. Khovanov and Lauda showed that $\mathrm{Proj}(H)$ also provides a categorification of $\mathbf{f}$: there is a unique algebra isomorphism

$$\gamma : \mathbf{f} \xrightarrow{\sim} \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\mathrm{Proj}(H)], \qquad \theta_i \mapsto [H_{\alpha_i}],$$

where θ_i is the generator of $\mathbf{f}$ corresponding to simple root α_i. In simply-laced types for $\mathbb{K}$ of characteristic zero, Rouquier [R2] and Varagnolo and Vasserot [VV] have shown further that this algebraic categorification of $\mathbf{f}$ is equivalent to Lusztig's geometric one; in particular γ maps the canonical basis of $\mathbf{f}$ to the basis for $[\mathrm{Proj}(H)]$ arising from the isomorphism classes of graded self-dual indecomposable projective modules.

The setup can also be dualized. Let $\mathrm{Rep}(H)$ be the abelian category of all finite dimensional graded left H-modules. Its Grothendieck group $[\mathrm{Rep}(H)]$ is again a $\mathbb{Z}[q,q^{-1}]$-algebra. Taking a dual map to γ yields another algebra isomorphism

$$\gamma^* : \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\mathrm{Rep}(H)] \xrightarrow{\sim} \mathbf{f}.$$

In simply-laced types with $\mathrm{char}\,\mathbb{K} = 0$, this sends the basis for $[\mathrm{Rep}(H)]$ arising from isomorphism classes of graded self-dual irreducible H-modules to the dual canonical basis for $\mathbf{f}$. In general, the graded self-dual irreducible H-modules still yield a basis for $\mathbf{f}$, but this basis can be different from the dual canonical basis; for an example in type G_2 in characteristic zero see [T]; for an example in type A in positive characteristic see [Wi] and also Example 2.16 below. However several people (e.g. [KOP]) have observed that it is always a *perfect basis* in the sense of Berenstein and Kazhdan [BeK]. This implies for any ground field that the irreducible H-modules are parametrized in a canonical way by Kashiwara's crystal $B(\infty)$ associated to $\mathbf{f}$, a result established originally by Lauda and Vazirani [LV] without using the theory of perfect bases.

Using the geometric approach of Varagnolo and Vasserot, hence for simply-laced types over fields of characteristic zero only, Kato [Ka] has explained further how to lift the PBW and dual PBW bases of $\mathbf{f}$ to certain graded modules $\{\widetilde{E}_b \mid b \in B(\infty)\}$ and $\{E_b \mid b \in B(\infty)\}$ over KLR algebras. We refer to these modules as *standard* and *proper standard* modules, respectively, motivated by the similarity to the theory of properly stratified algebras [D]. Kato establishes that each proper standard module E_b has irreducible head L_b, and the modules $\{L_b \mid b \in B(\infty)\}$ give a complete set of graded self-dual irreducible H-modules. The standard module $\widetilde{E}_b$ is infinite dimensional, and should be viewed informally as a "maximal self-extension" of the finite dimensional proper standard module E_b. Kato's work shows in particular that each of the algebras H_α has finite global dimension.

More recently in [M], the third author has found a purely algebraic way to introduce proper standard modules, similar in spirit to the approach via Lyndon words developed in [KR2] but more general as it makes sense for an arbitrary choice of the convex ordering $\prec$. It produces in the end the same collection of proper standard modules as above but indexed instead by the set KP of *Kostant partitions*, i.e. non-increasing sequences $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l)$ of positive roots. Switching to this notation, we henceforth denote the proper standard module corresponding to λ by $\bar{\Delta}(\lambda)$. This module has irreducible head $L(\lambda)$, and the modules $\{L(\lambda) \mid \lambda \in \mathrm{KP}\}$ give a complete set of graded self-dual irreducible H-modules. Moreover there is a partial order $\leq$ on KP with respect to which the *decomposition matrix* $([\bar{\Delta}(\lambda) : L(\mu)])_{\lambda,\mu\in\mathrm{KP}}$ is unitriangular, i.e.

$$[\bar{\Delta}(\lambda) : L(\lambda)] = 1, \qquad [\bar{\Delta}(\lambda) : L(\mu)] = 0 \text{ for } \mu \not\leq \lambda.$$

All of this theory works also for non-simply-laced types and ground fields $\mathbb{K}$ of positive characteristic. Finally [M] gives a purely algebraic way to compute the global dimension of H_α: in all cases it is equal to the height of $\alpha \in Q^+$.

Letting $P(\lambda)$ denote the projective cover of $L(\lambda)$, the standard module $\Delta(\lambda)$ corresponding to λ may be defined as

$$\Delta(\lambda) := P(\lambda)\Big/ \sum_{\mu \not\leq \lambda} \sum_{f : P(\mu) \to P(\lambda)} \operatorname{im} f.$$

Taking graded duals, we also have the *costandard module* $\nabla(\lambda) := \Delta(\lambda)^{\circledast}$ and the *proper costandard module* $\bar{\nabla}(\lambda) := \bar{\Delta}(\lambda)^{\circledast}$. For simply-laced types in characteristic zero, Kato showed that these modules satisfy various homological properties familiar from the theory of quasi-hereditary algebras. Perhaps the most important of these is the following:

$$\operatorname{Ext}^d_H(\Delta(\lambda), \bar{\nabla}(\mu)) \cong \begin{cases} \mathbb{K} & \text{if } d = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

There are many pleasant consequences. For example, one can deduce that the projective module $P(\lambda)$ has a finite filtration with sections of the form $\Delta(\mu)$ and multiplicities satisfying *BGG reciprocity*:

$$[P(\lambda) : \Delta(\mu)] = [\bar{\Delta}(\mu) : L(\lambda)].$$

The main purpose of this article is to explain an elementary approach to the proof of these homological properties starting from the results of [M]. Our results apply to all finite types and fields $\mathbb{K}$ of arbitrary characteristic.

The basic idea is to exploit a different definition of the standard module $\Delta(\lambda)$. To start with, we construct *root modules* $\Delta(\alpha)$ for each $\alpha \in R^+$ by taking an inverse limit of some iterated self-extensions of the irreducible module $L(\alpha)$; these modules categorify Lusztig's root vectors $r_\alpha \in \mathbf{f}$. A key new observation is that the endomorphism algebra of a product $\Delta(\alpha)^{\circ m}$ of m copies of $\Delta(\alpha)$ is isomorphic to the *nil Hecke algebra NH_m*. Hence we can define the *divided power module* $\Delta(\alpha^m)$ by using a primitive idempotent in the nil Hecke algebra to project to an indecomposable direct summand of $\Delta(\alpha)^{\circ m}$. For $\lambda = (\gamma_1^{m_1}, \dots, \gamma_s^{m_s})$ with $\gamma_1 \succ \cdots \succ \gamma_s$ we then show that

$$\Delta(\lambda) \cong \Delta(\gamma_1^{m_1}) \circ \cdots \circ \Delta(\gamma_s^{m_s}),$$

and proceed to derive the homological properties by applying generalized Frobenius reciprocity; see Theorem 3.12 for the final result.

We also explain an alternative way to construct the root module $\Delta(\alpha)$ by induction on height. For simple α the root module $\Delta(\alpha)$ is just the regular module H_α. Then for a non-simple positive root α, we pick a *minimal pair* (β, γ) for α in the sense of [M] (in particular β and γ are positive roots summing to α) and show in Theorem 4.10 that there exists a short exact sequence

$$0 \longrightarrow q^{-\beta\cdot\gamma}\Delta(\beta) \circ \Delta(\gamma) \xrightarrow{\varphi} \Delta(\gamma) \circ \Delta(\beta) \longrightarrow [p_{\beta,\gamma} + 1]\Delta(\alpha) \longrightarrow 0,$$

where $p_{\beta,\gamma}$ is the largest integer p such that $\beta - p\gamma$ is a root, and $[n]$ denotes the quantum integer. The map φ in this short exact sequence is defined explicitly, so $\Delta(\alpha)$ could instead be *defined* recursively in terms of its cokernel.

For multiplicity-free positive roots (= all positive roots in type A) the above short exact sequences can be assembled into some explicit projective resolutions of the root modules, which can be viewed as a variation on the classical Koszul resolution from commutative algebra; see Theorem 4.12. The first non-trivial example comes from the highest root α in type A_3. Adopting the notation of Example A.1 below, our resolution of $\Delta(\alpha)$ in this special case is

$$0 \longrightarrow q^2 H_\alpha 1_{321} \xrightarrow{(-\tau_1\tau_2\ \ \tau_2)} qH_\alpha 1_{213} \oplus qH_\alpha 1_{312} \xrightarrow{\binom{\tau_1}{\tau_1\tau_2}} H_\alpha 1_{123} \longrightarrow 0,$$

where we view elements of the direct sum as row vectors and the maps are defined by right multiplication by the given matrices.

This article supersedes the preprint [BK] which considered simply-laced types only. We also point out that Theorem 4.7 below proves the length two conjecture formulated in [BK, Conjecture 2.16].

Conventions. By a module V over a $\mathbb{Z}$-graded algebra H, we *always* mean a graded left H-module. We write $\operatorname{rad} V$ (resp. $\operatorname{soc} V$) for the intersection of all maximal submodules (resp. the sum of all irreducible submodules) of V. We write q for the upwards degree shift functor: if $V = \bigoplus_{n\in\mathbb{Z}} V_n$ then qV has $(qV)_n := V_{n-1}$. More generally, given a formal Laurent series $f(q) = \sum_{n\in\mathbb{Z}} f_n q^n$ with coefficients $f_n \in \mathbb{N}$, $f(q)V$ denotes $\bigoplus_{n\in\mathbb{Z}} q^n V^{\oplus f_n}$. For modules U and V, we write $\hom_H(U, V)$ for homogeneous H-module homomorphisms, reserving $\operatorname{Hom}_H(U, V)$ for the graded vector space $\bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_H(U, V)_n$ where

$$\operatorname{Hom}_H(U, V)_n := \hom_H(q^n U, V) = \hom_H(U, q^{-n} V).$$

We define $\operatorname{ext}^d_H(U, V)$ and $\operatorname{Ext}^d_H(U, V)$ similarly. If V is a locally finite dimensional graded vector space, its graded dimension is

$$\operatorname{Dim} V := \sum_{n\in\mathbb{Z}} (\dim V_n) q^n.$$

For formal Laurent series $f(q) = \sum_{n\in\mathbb{Z}} f_n q^n$ and $g(q) = \sum_{n\in\mathbb{Z}} g_n q^n$, we write $f(q) \leq g(q)$ if $f_n \leq g_n$ for all $n \in \mathbb{Z}$. We need the following generality several times.

Lemma 1.1. *Let H be a $\mathbb{Z}$-graded algebra which is locally finite dimensional and bounded below. Suppose that $d > 0$ and U, V are finitely generated H-modules. If $\operatorname{ext}^d_H(U, L) = 0$ for all irreducible subquotients L of V, then $\operatorname{ext}^d_H(U, V) = 0$.*

Proof. If V is finite dimensional this is an easy induction exercise using the long exact sequence. Now assume that V is infinite dimensional. The assumptions imply that V has an exhaustive filtration $V = V_0 \supseteq V_1 \supseteq \cdots$ in which each V/V_r is finite dimensional; for example one can let V_r be the submodule generated by all homogeneous vectors of degree $\geq r$ in V. Then we have that $V = \varprojlim(V/V_r)$. By [W, Theorem 3.5.8], there is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 \operatorname{ext}^{d-1}_A(U, V/V_r) \longrightarrow \operatorname{ext}^d_A(U, V) \longrightarrow \varprojlim \operatorname{ext}^d_A(U, V/V_r) \longrightarrow 0.$$

The last term is zero, so we just need show that $\varprojlim^1 \operatorname{ext}^{d-1}_A(U, V/V_r) = 0$. This follows by [W, Proposition 3.5.7] if we can show that the tower $\left(\operatorname{ext}^{d-1}_H(U, V/V_r)\right)$ satisfies the Mittag-Leffler condition, i.e. the natural map $\operatorname{ext}^{d-1}_H(U, V/V_{r+1}) \to \operatorname{ext}^{d-1}_H(U, V/V_r)$ is surjective for each $r \geq 0$. But this follows on applying $\hom_H(U, -)$ to the short exact sequence $0 \to V_r/V_{r+1} \to V/V_{r+1} \to V/V_r \to 0$. $\square$

2. KLR ALGEBRAS

We begin by collecting some basic facts about the representation theory of finite type KLR algebras. The discussion of the contravariant form on proper standard modules in §2.6 is new.

2.1. **The twisted bialgebra f.** Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let R be the root system of $\mathfrak{g}$ with respect to some Cartan subalgebra, $R^+ \subset R$ be a set of positive roots, and $\{\alpha_i \mid i \in I\}$ be the corresponding simple roots. Let $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ be the *root lattice*, $Q^+ := \bigoplus_{i\in I} \mathbb{N}\alpha_i$, and define the *height* of $\alpha = \sum_{i\in I} c_i\alpha_i \in Q^+$ from $\mathrm{ht}(\alpha) := \sum_{i\in I} c_i$. The Killing form gives us a positive definite symmetric bilinear form

$$Q \times Q \to \mathbb{Z}, \qquad (\alpha,\beta) \mapsto \alpha \cdot \beta,$$

which we assume is normalized so that $d_i := \alpha_i \cdot \alpha_i/2$ is a positive integer for each $i \in I$ and $\sum_{i\in I} d_i$ as small as possible; in particular this means that $d_i = 1$ for all i if $\mathfrak{g}$ is simply-laced. Then for $\alpha \in R^+$ we let $d_\alpha := \alpha \cdot \alpha/2$. The *Cartan matrix* is the matrix $C = (c_{i,j})_{i,j\in I}$ defined from $c_{i,j} := \frac{1}{d_i}\alpha_i \cdot \alpha_j$. Finally we have the *Weyl group* W, which is the subgroup of $GL(Q)$ generated by the simple reflections $\{s_i \mid i \in I\}$ defined from $s_i(\beta) := \beta - \frac{1}{d_i}(\alpha_i \cdot \beta)\alpha_i$.

Now let q be an indeterminate and $\mathbb{A} := \mathbb{Q}(q)$. For $n \in \mathbb{Z}$ let $[n]$ be the quantum integer $(q^n - q^{-n})/(q - q^{-1})$. Assuming $n \geq 0$ let $[n]^! := [n][n-1]\cdots[1]$ be the quantum factorial. More generally, for $i \in I$ (resp. $\alpha \in R^+$) let $[n]_i$ and $[n]^!_i$ (resp. $[n]_\alpha$ and $[n]^!_\alpha$) denote the quantum integer and quantum factorial with q replaced by $q_i := q^{d_i}$ (resp. $q_\alpha := q^{d_\alpha}$). Let $\mathbf{f}$ be the free associative $\mathbb{A}$-algebra on generators $\{\theta_i \mid i \in I\}$ subject to the *quantum Serre relations*

$$\sum_{r+s=1-c_{i,j}} (-1)^r \theta_i^{(r)}\theta_j\theta_i^{(s)} = 0$$

for all $i, j \in I$ and $r \geq 1$, where $\theta_i^{(r)}$ denotes the *divided power* $\theta_i^r/[r]^!_i$.

There is a Q^+-grading $\mathbf{f} = \bigoplus_{\alpha\in Q^+} \mathbf{f}_\alpha$ defined so that θ_i is in degree α_i. Viewing $\mathbf{f}\otimes\mathbf{f}$ as an algebra with multiplication $(a\otimes b)(c\otimes d) := q^{-\beta\cdot\gamma}ac\otimes bd$ for $a \in \mathbf{f}_\alpha, b \in \mathbf{f}_\beta, c \in \mathbf{f}_\gamma$ and $d \in \mathbf{f}_\delta$, there is a unique algebra homomorphism

$$r : \mathbf{f} \to \mathbf{f}\otimes\mathbf{f}, \qquad \theta_i \mapsto \theta_i \otimes 1 + 1 \otimes \theta_i \tag{2.1}$$

making $\mathbf{f}$ into a twisted bialgebra. In his book [L2, §1.2.5, §33.1.2], Lusztig shows further that $\mathbf{f}$ possesses a unique non-degenerate symmetric bilinear form $(\cdot,\cdot)$ such that

$$(1,1) = 1, \qquad (\theta_i,\theta_j) = \frac{\delta_{i,j}}{1-q_i^2}, \qquad (ab,c) = (a\otimes b, r(c))$$

for all $i, j \in I$ and $a,b,c \in \mathbf{f}$; on the right hand side of the last equation $(\cdot,\cdot)$ is the product form on $\mathbf{f}\otimes\mathbf{f}$ defined from $(a\otimes b, c\otimes d) := (a,c)(b,d)$. Note here that our q is Lusztig's v^{-1}.

Let $\mathscr{A} := \mathbb{Z}[q,q^{-1}] \subset \mathbb{A}$. Lusztig's $\mathscr{A}$-form $\mathbf{f}_\mathscr{A}$ for $\mathbf{f}$ is the $\mathscr{A}$-subalgebra of $\mathbf{f}$ generated by all $\theta_i^{(r)}$. Also let $\mathbf{f}^*_\mathscr{A}$ be the dual of $\mathbf{f}_\mathscr{A}$ with respect to the form $(\cdot,\cdot)$, i.e. $\mathbf{f}^*_\mathscr{A} := \left\{y \in \mathbf{f} \,\middle|\, (x,y) \in \mathscr{A} \text{ for all } x \in \mathbf{f}_\mathscr{A}\right\}$. It is another $\mathscr{A}$-subalgebra of $\mathbf{f}$. Moreover, both $\mathbf{f}_\mathscr{A}$ and $\mathbf{f}^*_\mathscr{A}$ are free as $\mathscr{A}$-modules, and we can identify

$$\mathbf{f} = \mathbb{A}\otimes_\mathscr{A} \mathbf{f}_\mathscr{A} = \mathbb{A}\otimes_\mathscr{A} \mathbf{f}^*_\mathscr{A}.$$

The field $\mathbb{A}$ possesses a unique automorphism called the *bar involution* such that $\overline{q} = q^{-1}$. With respect to this involution, let $\mathsf{b} : \mathbf{f} \to \mathbf{f}$ be the anti-linear algebra automorphism such that $\mathsf{b}(\theta_i) = \theta_i$ for all $i \in I$. Also let $\mathsf{b}^* : \mathbf{f} \to \mathbf{f}$ be the adjoint anti-linear map to b with respect to Lusztig's form, so b^* is defined from $(x, \mathsf{b}^*(y)) = \overline{(\mathsf{b}(x), y)}$ for any $x, y \in \mathbf{f}$. The maps b and b^* preserve $\mathbf{f}_{\mathscr{A}}$ and $\mathbf{f}^*_{\mathscr{A}}$, respectively.

Next let $\langle I \rangle$ be the free monoid on I, that is, the set of all *words* $\boldsymbol{i} = i_1 \cdots i_n$ for $n \geq 0$ and $i_1, \dots, i_n \in I$ with multiplication given by concatenation of words. For a word $\boldsymbol{i} = i_1 \cdots i_n$ of length n and a permutation $w \in S_n$, we let

$$|\boldsymbol{i}| := \alpha_{i_1} + \cdots + \alpha_{i_n}, \qquad w(\boldsymbol{i}) := i_{w^{-1}(1)} \cdots i_{w^{-1}(n)},$$
$$\theta_{\boldsymbol{i}} := \theta_{i_1} \cdots \theta_{i_n}, \qquad \deg(w; \boldsymbol{i}) := -\sum_{\substack{1 \leq j < k \leq n \\ w(j) > w(k)}} \alpha_{i_j} \cdot \alpha_{i_k}.$$

Setting $\langle I \rangle_\alpha := \{\boldsymbol{i} \in \langle I \rangle \mid |\boldsymbol{i}| = \alpha\}$, the monomials $\{\theta_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$ span $\mathbf{f}_\alpha$. The *quantum shuffle algebra* is the free $\mathscr{A}$-module $\mathscr{A}\langle I \rangle = \bigoplus_{\alpha \in Q^+} \mathscr{A}\langle I \rangle_\alpha$ on basis $\langle I \rangle$, viewed as an $\mathscr{A}$-algebra via the *shuffle product* $\circ$ defined on words $\boldsymbol{i}$ and $\boldsymbol{j}$ of lengths m and n, respectively, by

$$\boldsymbol{i} \circ \boldsymbol{j} := \sum_{\substack{w \in S_{m+n} \\ w(1) < \cdots < w(m) \\ w(m+1) < \cdots < w(m+n)}} q^{\deg(w; \boldsymbol{i}\boldsymbol{j})} w(\boldsymbol{i}\boldsymbol{j}). \tag{2.2}$$

As observed originally by Green and Rosso, there is an injective $\mathscr{A}$-algebra homomorphism

$$\mathrm{Ch} : \mathbf{f}^*_{\mathscr{A}} \to \mathscr{A}\langle I \rangle, \qquad x \mapsto \sum_{\boldsymbol{i} \in \langle I \rangle} (\theta_{\boldsymbol{i}}, x) \boldsymbol{i}. \tag{2.3}$$

This intertwines the anti-linear involution b^* on $\mathbf{f}^*_{\mathscr{A}}$ with the bar involution on $\mathscr{A}\langle I \rangle$, which is defined from $\overline{\sum_{\boldsymbol{i} \in \langle I \rangle} a_{\boldsymbol{i}} \boldsymbol{i}} := \sum_{\boldsymbol{i} \in \langle I \rangle} \overline{a}_{\boldsymbol{i}} \boldsymbol{i}$. Using (2.2), one checks further that $\overline{\boldsymbol{i} \circ \boldsymbol{j}} = q^{|\boldsymbol{i}| \cdot |\boldsymbol{j}|} \boldsymbol{j} \circ \boldsymbol{i}$. Hence

$$\mathsf{b}^*(xy) = q^{\beta \cdot \gamma} \mathsf{b}^*(y) \mathsf{b}^*(x) \tag{2.4}$$

for $x \in \mathbf{f}_\beta$ and $y \in \mathbf{f}_\gamma$. Using this and induction on height, it follows that

$$\mathsf{b}^*(x) = (-1)^n q^{d_\alpha + d_{i_1} + \cdots + d_{i_n}} \mathsf{b}(\sigma(x)) \tag{2.5}$$

for $x \in \mathbf{f}_\alpha$ and $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_n} \in Q^+$, where $\sigma : \mathbf{f} \to \mathbf{f}$ is the algebra anti-automorphism such that $\sigma(\theta_i) = \theta_i$ for each $i \in I$.

2.2. **The KLR algebra.** Fix now a field $\mathbb{K}$. Also choose signs $\varepsilon_{i,j}$ for all $i, j \in I$ with $c_{i,j} < 0$ so that $\varepsilon_{i,j}\varepsilon_{j,i} = -1$. For $\alpha \in Q^+$ of height n, the *KLR algebra* H_α is the associative, unital $\mathbb{K}$-algebra defined by generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\} \cup \{x_1, \dots, x_n\} \cup \{\tau_1, \dots, \tau_{n-1}\}$$

subject only to the following relations:

- $x_k x_l = x_l x_k$;
- the elements $\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\}$ are mutually orthogonal idempotents whose sum is the identity $1_\alpha \in H_\alpha$;
- $x_k 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} x_k$ and $\tau_k 1_{\boldsymbol{i}} = 1_{(k\,k+1)(\boldsymbol{i})} \tau_k$;
- $(\tau_k x_l - x_{(k\,k+1)(l)} \tau_k) 1_{\boldsymbol{i}} = \begin{cases} 1_{\boldsymbol{i}} & \text{if } i_k = i_{k+1} \text{ and } l = k+1, \\ -1_{\boldsymbol{i}} & \text{if } i_k = i_{k+1} \text{ and } l = k, \\ 0 & \text{otherwise;} \end{cases}$

- $\tau_k^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_k = i_{k+1}, \\ \varepsilon_{i_k,i_{k+1}}(x_k{}^{-c_{i_k,i_{k+1}}} - x_{k+1}{}^{-c_{i_{k+1},i_k}})1_{\boldsymbol{i}} & \text{if } c_{i_k,i_{k+1}} < 0, \\ 1_{\boldsymbol{i}} & \text{otherwise;} \end{cases}$
- $\tau_k \tau_l = \tau_l \tau_k$ if $|k - l| > 1$;
- $(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)1_{\boldsymbol{i}} =$
$$\begin{cases} \displaystyle\sum_{r+s=-1-c_{i_k,i_{k+1}}} \varepsilon_{i_k,i_{k+1}} x_k^r x_{k+2}^s 1_{\boldsymbol{i}} & \text{if } c_{i_k,i_{k+1}} < 0 \text{ and } i_k = i_{k+2}, \\ 0 & \text{otherwise.} \end{cases}$$

The algebra H_α is $\mathbb{Z}$-graded with $1_{\boldsymbol{i}}$ in degree zero, $x_k 1_{\boldsymbol{i}}$ in degree $2d_{i_k}$ and $\tau_k 1_{\boldsymbol{i}}$ in degree $-\alpha_{i_k} \cdot \alpha_{i_{k+1}}$. There's also an anti-automorphism $T : H_\alpha \to H_\alpha$ which fixes all the generators.

Here are a few other basic facts about the structure of these algebras established in [KL1, KL2] or [R1]. Fix once and for all a reduced expression for each $w \in S_n$ and let τ_w be the corresponding product of the τ-generators of H_α. Note that $\tau_w 1_{\boldsymbol{i}}$ is of degree $\deg(w; \boldsymbol{i})$. The monomials

$$\{x_1^{k_1} \cdots x_n^{k_n} \tau_w 1_{\boldsymbol{i}} \mid w \in S_n, k_1, \dots, k_n \geq 0, \boldsymbol{i} \in \langle I \rangle_\alpha\} \tag{2.6}$$

give a basis for H_α. In particular, H_α is locally finite dimensional and bounded below. There is also an explicit description of the center $Z(H_\alpha)$, from which it follows that H_α is free of finite rank as a module over its center; forgetting the grading the rank is $(n!)^2$.

For $m \geq 1$ and $i \in I$, the KLR algebra $H_{m\alpha_i}$ is identified with the *nil Hecke algebra* NH_m, that is, the algebra with generators $x_1, \dots, x_m$ and $\tau_1, \dots, \tau_{m-1}$ subject to the following relations: $x_i x_j = x_j x_i$; $\tau_i x_j = x_j \tau_i$ for $j \neq i, i+1$; $\tau_i x_{i+1} = x_i \tau_i + 1$; $x_{i+1}\tau_i = \tau_i x_i + 1$; $\tau_i^2 = 0$; and the usual type A braid relations amongst $\tau_1, \dots, \tau_{m-1}$. It is well known that the nil Hecke algebra is a matrix algebra over its center; see e.g. [R2, §2] for a recent exposition. Moreover, writing $w_{[1,m]}$ for the longest element of S_m, the degree zero element

$$e_m := x_2 x_3^2 \cdots x_m^{m-1} \tau_{w_{[1,m]}} \tag{2.7}$$

is a primitive idempotent, hence $P(\alpha_i^m) := q_i^{\frac{1}{2}m(m-1)} H_{m\alpha_i} e_m$ is an indecomposable projective $H_{m\alpha_i}$-module. The degree shift here has been chosen so that irreducible head $L(\alpha_i^m)$ of $P(\alpha_i^m)$ has graded dimension $[m]_i^!$. Thus $H_{m\alpha_i} \cong [m]_i^! P(\alpha_i^m)$ as a left module.

For $\beta, \gamma \in Q^+$, there is an evident non-unital algebra embedding $H_\beta \otimes H_\gamma \hookrightarrow H_{\beta+\gamma}$. We denote the image of the identity $1_\beta \otimes 1_\gamma \in H_\beta \otimes H_\gamma$ by $1_{\beta,\gamma} \in H_{\beta+\gamma}$. Then for an $H_{\beta+\gamma}$-module U and an $H_\beta \otimes H_\gamma$-module V, we set

$$\mathrm{res}^{\beta+\gamma}_{\beta,\gamma} U := 1_{\beta,\gamma} U, \qquad \mathrm{ind}^{\beta+\gamma}_{\beta,\gamma} V := H_{\beta+\gamma} 1_{\beta,\gamma} \otimes_{H_\beta \otimes H_\gamma} V,$$

which are naturally $H_\beta \otimes H_\gamma$- and $H_{\beta+\gamma}$-modules, respectively. These definitions extend in an obvious way to situations where there are more than two tensor factors. The following Mackey-type theorem is of crucial importance.

Theorem 2.1. *Suppose we are given $\beta, \gamma, \beta', \gamma' \in Q^+$ of heights m, n, m', n', respectively, such that $\beta + \gamma = \beta' + \gamma'$. Setting $k := \min(m, n, m', n')$, let $\{1 = w_0 < \cdots < w_k\}$ be the set of minimal length $S_{m'} \times S_{n'} \backslash S_{m+n} / S_m \times S_n$-double coset representatives ordered via the Bruhat order. For any $H_\beta \otimes H_\gamma$-module V, there is a filtration*

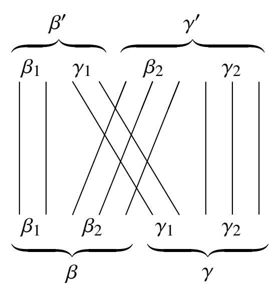
$$0 = V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots \subseteq V_k = \mathrm{res}^{\beta'+\gamma'}_{\beta',\gamma'} \circ \mathrm{ind}^{\beta+\gamma}_{\beta,\gamma}(V)$$

defined by $V_j := \sum_{i=0}^{j} \sum_{w \in (S_{m'} \times S_{n'}) w_i (S_m \times S_n)} 1_{\beta',\gamma'} \tau_w 1_{\beta,\gamma} \otimes V$. *Moreover there is a unique isomorphism of* $H_{\beta'} \otimes H_{\gamma'}$*-modules*

$$V_j / V_{j-1} \xrightarrow{\sim} \bigoplus_{\beta_1,\beta_2,\gamma_1,\gamma_2} q^{-\beta_2 \cdot \gamma_1} \mathrm{ind}^{\beta',\gamma'}_{\beta_1,\gamma_1,\beta_2,\gamma_2} \circ I^* \circ \mathrm{res}^{\beta,\gamma}_{\beta_1,\beta_2,\gamma_1,\gamma_2}(V),$$

$$1_{\beta',\gamma'} \tau_{w_j} 1_{\beta,\gamma} \otimes v + V_{j-1} \mapsto \sum_{\beta_1,\beta_2,\gamma_1,\gamma_2} 1_{\beta_1,\gamma_1,\beta_2,\gamma_2} \otimes 1_{\beta_1,\beta_2,\gamma_1,\gamma_2} v,$$

where $I : H_{\beta_1} \otimes H_{\gamma_1} \otimes H_{\beta_2} \otimes H_{\gamma_2} \xrightarrow{\sim} H_{\beta_1} \otimes H_{\beta_2} \otimes H_{\gamma_1} \otimes H_{\gamma_2}$ *is the obvious isomorphism, and the sums are taken over all* $\beta_1, \beta_2, \gamma_1, \gamma_2 \in Q^+$ *such that* $\beta_1 + \beta_2 = \beta, \gamma_1 + \gamma_2 = \gamma, \beta_1 + \gamma_1 = \beta', \beta_2 + \gamma_2 = \gamma'$ *and* $\min(\mathrm{ht}(\beta_2), \mathrm{ht}(\gamma_1)) = j$*:*

$w_j =$ 

Proof. This follows as in [KL1, Proposition 2.18]. □

2.3. **The categorification theorem.** Let $\mathrm{Rep}(H_\alpha)$ denote the abelian category of finite dimensional H_α-modules and set

$$\mathrm{Rep}(H) := \bigoplus_{\alpha \in Q^+} \mathrm{Rep}(H_\alpha).$$

This is a graded $\mathbb{K}$-linear monoidal category with respect to the induction product $U \circ V := \mathrm{ind}^{\beta+\gamma}_{\beta,\gamma}(U \boxtimes V)$ for $U \in \mathrm{Rep}(H_\beta)$ and $V \in \mathrm{Rep}(H_\gamma)$. Let $[\mathrm{Rep}(H)] = \bigoplus_{\alpha \in Q^+} [\mathrm{Rep}(H_\alpha)]$ denote its Grothendieck ring, which we make into an $\mathscr{A}$-algebra so that $q[V] = [qV]$. Dually, we have the additive category $\mathrm{Proj}(H_\alpha)$ of finitely generated projective H_α-modules and set

$$\mathrm{Proj}(H) := \bigoplus_{\alpha \in Q^+} \mathrm{Proj}(H_\alpha).$$

Again this is a graded $\mathbb{K}$-linear monoidal category with respect to the induction product, and again the split Grothendieck group $[\mathrm{Proj}(H)] = \bigoplus_{\alpha \in Q^+} [\mathrm{Proj}(H_\alpha)]$ is naturally an $\mathscr{A}$-algebra. Moreover there is a non-degenerate pairing

$$(\cdot, \cdot) : [\mathrm{Proj}(H)] \times [\mathrm{Rep}(H)] \to \mathscr{A}$$

defined on $P \in \mathrm{Proj}(H_\alpha)$ and $V \in \mathrm{Rep}(H_\beta)$ by declaring that

$$([P], [V]) := \begin{cases} \mathrm{Dim}\, T^*(P) \otimes_{H_\alpha} V & \text{if } \beta = \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $T^*(P)$ denotes P viewed as a right module via the anti-automorphism T. Finally there are dualities $\circledast$ on $\mathrm{Rep}(H_\alpha)$ and $\#$ on $\mathrm{Proj}(H_\alpha)$ inducing antilinear involutions on the Grothendieck groups. These are defined from $V^\circledast := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and $P^\# := \mathrm{Hom}_{H_\alpha}(P, H_\alpha)$, respectively, both viewed as left modules via T; more generally $V^\circledast$ makes good sense for any V that is locally finite dimensional.

Theorem 2.2 (Khovanov-Lauda)**.** *There is a unique adjoint pair of $\mathscr{A}$-algebra isomorphisms*

$$\gamma : \mathbf{f}_{\mathscr{A}} \xrightarrow{\sim} [\mathrm{Proj}(H)], \qquad \gamma^* : [\mathrm{Rep}(H)] \xrightarrow{\sim} \mathbf{f}^*_{\mathscr{A}}$$

such that $\gamma(\theta_i^{(n)}) = [P(\alpha_i^n)]$. Under these isomorphisms, the antilinear involutions b *and* b^* *on $\mathbf{f}_{\mathscr{A}}$ and $\mathbf{f}^*_{\mathscr{A}}$ correspond to the dualities # and $\circledast$, respectively.*

Proof. See [KL1, §3] and [KL2, Theorem 8] for the statement about γ. The dual statement is implicit in [KL1]; see also [KR2, Theorem 4.4]. □

Henceforth we will *identify* $\mathbf{f}_{\mathscr{A}}$ with $[\mathrm{Proj}(H)]$ and $\mathbf{f}^*_{\mathscr{A}}$ with $[\mathrm{Rep}(H)]$ according to Theorem 2.2. Any H_α-module V admits a decomposition into *word spaces* $V = \bigoplus_{\boldsymbol{i}\in\langle I\rangle_\alpha} 1_{\boldsymbol{i}} V$. Then the *character* of $V \in \mathrm{Rep}(H_\alpha)$ is the formal sum

$$\mathrm{Ch}\, V = \sum_{\boldsymbol{i}\in\langle I\rangle_\alpha} (\mathrm{Dim}\, 1_{\boldsymbol{i}} V)\boldsymbol{i} \in \mathscr{A}\langle I\rangle_\alpha. \tag{2.8}$$

As $(\theta_{\boldsymbol{i}}, [V]) = ([H_\alpha 1_{\boldsymbol{i}}], [V]) = \mathrm{Dim}\, 1_{\boldsymbol{i}} H_\alpha \otimes_{H_\alpha} V = \mathrm{Dim}\, 1_{\boldsymbol{i}} V$, we have that $\mathrm{Ch}\, V = \mathrm{Ch}[V]$, where Ch on the right hand side is the injective map from (2.3). Also note the following, which is the module-theoretic analogue of (2.4).

Lemma 2.3. *For $U \in \mathrm{Rep}(H_\beta)$ and $V \in \mathrm{Rep}(H_\gamma)$, there is a natural isomorphism $(U \circ V)^\circledast \cong q^{\beta\cdot\gamma} V^\circledast \circ U^\circledast$.*

Proof. This is [LV, Theorem 2.2(2)]. □

2.4. **PBW and dual PBW bases.** A *convex ordering* on R^+ is a total order $\prec$ such that

$$\beta, \gamma, \beta+\gamma \in R^+, \beta \prec \gamma \quad \Rightarrow \quad \beta \prec \beta+\gamma \prec \gamma.$$

By [P], there is a bijection between convex orderings of R^+ and reduced expressions for the longest element w_0 of W: given a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ the corresponding convex ordering on R^+ is given by

$$\alpha_{i_1} \prec s_{i_1}(\alpha_{i_2}) \prec s_{i_1} s_{i_2}(\alpha_{i_3}) \prec \cdots \prec s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N}).$$

We assume henceforth that such a convex ordering/reduced expression has been specified. The following lemma is very useful.

Lemma 2.4. *Suppose we are given positive roots $\alpha, \beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_l$ such that $\beta_i \preceq \alpha \preceq \gamma_j$ for all i and j. We have that $\beta_1 + \cdots + \beta_k = \gamma_1 + \cdots + \gamma_l$ if and only if $k = l$ and $\beta_1 = \cdots = \beta_k = \gamma_1 = \cdots = \gamma_l = \alpha$.*

Proof. Suppose that $\beta_1 + \cdots + \beta_k = \gamma_1 + \cdots + \gamma_l$. We may assume for suitable $0 \le k' \le k$ and $0 \le l' \le l$ that $\beta_i = \alpha$ for $1 \le i \le k'$, $\beta_i \prec \alpha$ for $k'+1 \le i \le k$ and $\gamma_i = \alpha$ for $1 \le i \le l'$, $\gamma_i \succ \alpha$ for $l'+1 \le i \le l$. Then we need to show that $k = k' = l' = l$. Assume the convex ordering corresponds to reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ as above. Then $\alpha = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for a unique $1 \le j \le N$. If $k' \ge l'$, let $w := s_{i_j} \cdots s_{i_1}$. From $\beta_1 + \cdots + \beta_k = \gamma_1 + \cdots + \gamma_l$, we deduce that

$$(k'-l')w(\alpha) + w(\beta_{k'+1}) + \cdots + w(\beta_k) = w(\gamma_{l'+1}) + \cdots + w(\gamma_l).$$

By [Bo, Ch. VI, §6, Cor. 2], the set of positive roots sent to negative roots by w is the set $\{\alpha' \in R^+ \mid \alpha' \preceq \alpha\}$. Hence the left hand side of the above equation is a sum of negative roots and the right hand side is a sum of positive roots. So both sides are zero and we deduce that $k = k' = l' = l$. For the case $k' \le l'$, argue in a similar way with $w := s_{i_{j-1}} \cdots s_{i_1}$, so that the set of positive roots sent to negative by w is $\{\alpha' \in R^+ \mid \alpha' \prec \alpha\}$. $\square$

Corresponding to the chosen convex ordering/reduced expression, Lusztig has introduced *root vectors* $\{r_\alpha \mid \alpha \in R^+\}$ in $\mathbf{f}$ via a certain braid group action. The definition uses the embedding of $\mathbf{f}$ into the full quantum group $U_q(\mathfrak{g})$ so we only summarize it briefly: we take the positive embedding $\mathbf{f} \hookrightarrow U_q(\mathfrak{g}), x \mapsto x^+$ defined from $\theta_i^+ := E_i$ and use the braid group generators $T_i := T''_{i,+}$ from [L2, §37.1.3] (recalling our q is Lusztig's v^{-1}); then for $\alpha \in R^+$ the root element r_α is the unique element of $\mathbf{f}$ such that

$$r_\alpha^+ = T_{i_1} \cdots T_{i_{j-1}}(E_{i_j})$$

if $\alpha = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$. For example, in type A_2 with $I = \{1, 2\}$ and fixed reduced expression $w_0 = s_1 s_2 s_1$, so that $\alpha_1 \prec \alpha_1 + \alpha_2 \prec \alpha_2$, we have that $r_{\alpha_1} = \theta_1, r_{\alpha_1+\alpha_2} = \theta_1\theta_2 - q\theta_2\theta_1, r_{\alpha_2} = \theta_2$. Also introduce the *dual root vector*

$$r_\alpha^* := (1 - q_\alpha^2) r_\alpha. \tag{2.9}$$

The normalization here ensures that r_α^* is invariant under b^*, as can be checked directly using (2.5) and the formulae in [L2, §37.2.4].

A *Kostant partition* of $\alpha \in Q^+$ is a sequence $\lambda = (\lambda_1, \dots, \lambda_l)$ of positive roots such that $\lambda_1 \succeq \cdots \succeq \lambda_l$ and $\lambda_1 + \cdots + \lambda_l = \alpha$. Denote the set of all Kostant partitions of α by $\mathrm{KP}(\alpha)$. For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$, let $m_\beta(\lambda)$ denote the multiplicity of $\beta \in R^+$ as a part of λ. Also set $\lambda'_k := \lambda_{l+1-k}$ for $k = 1, \dots, l$. Then define a partial order $\leq$ on $\mathrm{KP}(\alpha)$ so that $\lambda < \mu$ if and only if both of the following hold:

- $\lambda_1 = \mu_1, \dots, \lambda_{k-1} = \mu_{k-1}$ and $\lambda_k \prec \mu_k$ for some k such that λ_k and μ_k both make sense;
- $\lambda'_1 = \mu'_1, \dots, \lambda'_{k-1} = \mu'_{k-1}$ and $\lambda'_k \succ \mu'_k$ for some k such that λ'_k and μ'_k both make sense.

This ordering was introduced in [M, §3], and the following lemmas were noted already there (at least implicitly).

Lemma 2.5. *For $\alpha \in R^+$ and $m \geq 1$, the Kostant partition (α^m) is the unique smallest element of* $\mathrm{KP}(m\alpha)$.

Proof. Suppose that $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(m\alpha)$ satisfies $\lambda \not> (\alpha^m)$. Then we either have that $\lambda_1 \preceq \alpha$ or that $\lambda'_1 \succeq \alpha$. In the former case, $\lambda_k \preceq \alpha$ for all k, while in the latter $\lambda_k \succeq \alpha$ for all k. Either way, applying Lemma 2.4 to the equality $\lambda_1 + \cdots + \lambda_l = \alpha + \cdots + \alpha$ (m times), we deduce that $\lambda = (\alpha^m)$. $\square$

Lemma 2.6. *For $\alpha \in R^+$, suppose that $\lambda \in \mathrm{KP}(\alpha)$ is minimal such that $\lambda > \alpha$. Then λ has two parts, i.e. $\lambda = (\beta, \gamma)$ for positive roots $\beta \succ \alpha \succ \gamma$.*

Proof. Suppose for a contradiction that $\lambda = (\lambda_1, \dots, \lambda_l)$ with $l \geq 3$. By [M, Lemma 2.1], we can partition the set $\{1, \dots, l\}$ as $J \sqcup K$ so that $\beta := \sum_{j \in J} \lambda_j$ and $\gamma := \sum_{k \in K} \lambda_k$ are positive roots with $\beta \succ \gamma$. Each λ_j is $\preceq \lambda_1$ hence by Lemma 2.4 we have that $\beta \preceq \lambda_1$. Moreover if it happens that $\beta = \lambda_1$ then $\gamma = \alpha - \beta = \lambda_2 + \cdots + \lambda_l$ and we see similarly that $\gamma \preceq \lambda_2$. As $l \geq 3$, this shows that either $\beta \prec \lambda_1$, or $\beta = \lambda_1$ and $\gamma \prec \lambda_2$. A similar argument shows that either $\gamma \succ \lambda'_1$, or $\gamma = \lambda'_1$ and $\beta \succ \lambda'_2$. Hence $(\beta, \gamma) < \lambda$. But also we know that $(\beta, \gamma) > (\alpha)$ by Lemma 2.5. So this contradicts the minimality of λ. $\square$

Let $\mathrm{KP} := \bigcup_{\alpha \in Q^+} \mathrm{KP}(\alpha)$. For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}$, we set

$$r_\lambda := r_{\lambda_1} \cdots r_{\lambda_l} / [\lambda]^!, \qquad r^*_\lambda := q^{s_\lambda} r^*_{\lambda_1} \cdots r^*_{\lambda_l}, \tag{2.10}$$

where

$$[\lambda]^! := \prod_{\beta \in R^+} [m_\beta(\lambda)]^!_\beta, \qquad s_\lambda := \sum_{\beta \in R^+} \frac{d_\beta}{2} m_\beta(\lambda)(m_\beta(\lambda) - 1).$$

The following key result is due to Lusztig; it gives us the *PBW* and *dual PBW bases* for $\mathbf{f}$ arising from the given convex ordering $\prec$.

Theorem 2.7 (Lusztig)**.** *The monomials* $\{r_\lambda \mid \lambda \in \mathrm{KP}\}$ *and* $\left\{r^*_\lambda \mid \lambda \in \mathrm{KP}\right\}$ *give a pair of dual bases for the free* $\mathscr{A}$*-modules* $\mathbf{f}_\mathscr{A}$ *and* $\mathbf{f}^*_\mathscr{A}$*, respectively.*

Proof. This follows from [L2, Corollary 41.1.4(b)], [L2, Proposition 41.1.7], [L2, Proposition 38.2.3] and [L2, Lemma 1.2.8(b)]. □

2.5. **Proper standard modules.** The next results are taken from [M, §3], which generalizes [KR2]. Note that our conventions for the ordering $\preceq$ are consistent with the notation in [KR2]; the ordering in [M] is the opposite of the ordering here. The modules $L(\alpha)$ in the following theorem are called *cuspidal modules* in [KR2, M].

Theorem 2.8. *For* $\alpha \in R^+$ *there is a unique (up to isomorphism) irreducible* H_α*-module* $L(\alpha)$ *such that* $[L(\alpha)] = r^*_\alpha$*. Moreover, for any* $m \geq 1$*, the module* $L(\alpha^m) := q_\alpha^{\frac{1}{2}m(m-1)} L(\alpha)^{\circ m}$ *is irreducible.*

Proof. The existence of $L(\alpha)$ is the first part of [M, Theorem 3.1]. The second part is [M, Lemma 3.4]. □

Suppose we are given $\alpha \in Q^+$ and $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$. Define the *proper standard module*

$$\bar{\Delta}(\lambda) := q^{s_\lambda} L(\lambda_1) \circ \cdots \circ L(\lambda_l). \tag{2.11}$$

It is immediate from Theorem 2.8 and the definition (2.10) that $[\bar{\Delta}(\lambda)] = r^*_\lambda$, i.e. proper standard modules categorify the dual PBW basis. Let

$$L(\lambda) := \bar{\Delta}(\lambda) / \mathrm{rad}\, \bar{\Delta}(\lambda).$$

The following theorem asserts in particular that this is a self-dual irreducible module.

Theorem 2.9. *For* $\alpha \in Q^+$ *the modules* $\{L(\lambda) \mid \lambda \in \mathrm{KP}(\alpha)\}$ *give a complete set of pairwise inequivalent* $\circledast$*-self-dual irreducible* H_α*-modules. Moreover, for any* $\lambda \in \mathrm{KP}(\alpha)$*, all composition factors of* $\mathrm{rad}\, \bar{\Delta}(\lambda)$ *are of the form* $q^n L(\mu)$ *for* $\mu \prec \lambda$ *and* $n \in \mathbb{Z}$*.*

Proof. This is [M, Theorem 3.1]. □

For $\lambda \in \mathrm{KP}$, we denote the projective cover of $L(\lambda)$ by $P(\lambda)$. Also introduce the *proper costandard module*

$$\bar{\nabla}(\lambda) := \bar{\Delta}(\lambda)^\circledast. \tag{2.12}$$

It is immediate from Theorem 2.9 that $\bar{\nabla}(\lambda)$ has socle $L(\lambda)^\circledast \cong L(\lambda)$. Let us also record the key lemma (known by the first two authors as "McNamara's Lemma") at the heart of the proof of both of the above theorems.

Lemma 2.10. *Suppose we are given* $\alpha \in R^+$ *and* $\beta, \gamma \in Q^+$ *with* $\beta + \gamma = \alpha$*. If* $\mathrm{res}^\alpha_{\beta,\gamma} L(\alpha) \neq 0$ *then* β *is a sum of positive roots* $\preceq \alpha$ *and* γ *is a sum of positive roots* $\succeq \alpha$*.*

Proof. This is [M, Lemma 3.2]. □

Here are some further consequences.

Lemma 2.11. *For $\alpha \in R^+$ and $m \geq 1$, we have that*

$$\left[\mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha} L(\alpha^m)\right] = [m]^!_\alpha \left[L(\alpha)^{\boxtimes m}\right].$$

Proof. It suffices to show for $m \geq 2$ that $\left[\mathrm{res}^{m\alpha}_{\alpha,(m-1)\alpha} L(\alpha^m)\right] = [m]_\alpha [L(\alpha) \boxtimes L(\alpha^{m-1})]$. For this we apply Theorem 2.1, noting that $L(\alpha^m) = q_\alpha^{(m-1)} L(\alpha) \circ L(\alpha^{m-1})$. To understand the non-zero layers in the Mackey filtration, we need to find all quadruples $(\beta_1, \beta_2, \gamma_1, \gamma_2)$ such that $\beta_1 + \beta_2 = \beta_1 + \gamma_1 = \alpha, \gamma_1 + \gamma_2 = \beta_2 + \gamma_2 = (m-1)\alpha$, $\mathrm{res}^\alpha_{\beta_1,\beta_2} L(\alpha) \neq 0$ and $\mathrm{res}^{(m-1)\alpha}_{\gamma_1,\gamma_2} L(\alpha^{m-1}) \neq 0$. By Lemma 2.10 and Mackey, both β_1 and γ_1 are sums of positive roots $\leq \alpha$. Since $\beta_1 + \gamma_1 = \alpha$, we deduce using Lemma 2.4 that either $\beta_1 = 0$ or $\gamma_1 = 0$. This analysis shows that there are just two non-zero layers in the Mackey filtration. The bottom non-zero layer is obviously equal in the Grothendieck group to $q_\alpha^{(m-1)} \left[L(\alpha) \circ L(\alpha^{m-1})\right]$. Also using some induction on m, the top non-zero layer contributes $q_\alpha^{-1} [m-1]_\alpha \left[L(\alpha) \circ L(\alpha^{m-1})\right]$. Finally observe that $q_\alpha^{(m-1)} + q_\alpha^{-1}[m-1]_\alpha = [m]_\alpha$. □

Lemma 2.12. *Suppose we are given $\alpha \in Q^+$ and $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$. Let $\mathrm{res}^\alpha_\lambda$ denote the functor $\mathrm{res}^\alpha_{\lambda_1,\dots,\lambda_l}$. Then*

$$\left[\mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda)\right] = [\lambda]^! \left[L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l)\right],$$

Moreover for any $\mu \not\leq \lambda$ we have that $\mathrm{res}^\alpha_\mu \bar{\Delta}(\lambda) = 0$.

Proof. This follows from [M, Lemma 3.3] and Lemma 2.11. □

2.6. **The contravariant form and Williamson's counterexample.** The results in this subsection are not needed in the remainder of the article but are of independent interest. Throughout we fix $\alpha \in Q^+$ of height n.

Lemma 2.13. *For $\lambda \in \mathrm{KP}(\alpha)$ there is a unique (up to scalars) non-zero bilinear form $\langle \cdot, \cdot \rangle$ on $\bar{\Delta}(\lambda)$ such that*

$$\langle hv, v' \rangle = \langle v, T(h) v' \rangle \tag{2.13}$$

for all $v, v' \in \bar{\Delta}(\lambda)$ and $h \in H_\alpha$. The radical of this bilinear form is the unique maximal submodule of $\bar{\Delta}(\lambda)$. Moreover, for $\boldsymbol{i}, \boldsymbol{i}' \in \langle I \rangle_\alpha$ and $m, m' \in \mathbb{Z}$, we have that $\langle 1_{\boldsymbol{i}} \bar{\Delta}(\lambda)_m, 1_{\boldsymbol{i}'} \bar{\Delta}(\lambda)_{m'} \rangle = 0$ unless $\boldsymbol{i} = \boldsymbol{i}'$ and $m + m' = 0$.

Proof. There is an isomorphism from $\mathrm{Hom}_{H_\alpha}(\bar{\Delta}(\lambda), \bar{\nabla}(\lambda))$ to the space of bilinear forms on $\bar{\Delta}(\lambda)$ with the property (2.13), mapping $f : \bar{\Delta}(\lambda) \to \bar{\nabla}(\lambda)$ to the form $\langle v, v' \rangle := f(v)(v')$. Moreover $\mathrm{Hom}_{H_\alpha}(\bar{\Delta}(\lambda), \bar{\nabla}(\lambda))$ is one-dimensional, indeed, it is spanned by a homogeneous homomorphism that sends the head of $\bar{\Delta}(\lambda)$ onto the socle of $\bar{\nabla}(\lambda)$. The existence and uniqueness of the contravariant form follow at once. The last two parts of the lemma are immediate consequences too. □

The next lemma is useful when trying to compute the contravariant form on $\bar{\Delta}(\lambda)$ in practice. Recall for $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$ that

$$\bar{\Delta}(\lambda) = q^{s_\lambda} H_\alpha 1_\lambda \otimes_{H_\lambda} (L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l)),$$

where $H_\lambda := H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_l}$ with identity 1_λ. Letting S_λ denote the parabolic subgroup $S_{\mathrm{ht}(\lambda_1)} \times \cdots \times S_{\mathrm{ht}(\lambda_l)}$ of S_n and D_λ be the set of minimal length S_n/S_λ-coset representatives, any element of $\bar{\Delta}(\lambda)$ is a sum of vectors of the form $\tau_w 1_\lambda \otimes (v_1 \otimes \cdots \otimes v_l)$ for $w \in D_\lambda$ and $v_i \in L(\lambda_i)$.

Lemma 2.14. *For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$, let $x = x^{-1}$ be the longest element of D_λ such that $\tau_x 1_\lambda = 1_\lambda \tau_x$, and let $y = y^{-1}$ be the longest element of S_l such that $\lambda_{y(i)} = \lambda_i$ for each $i = 1, \dots, l$. The contravariant form $\langle \cdot, \cdot \rangle$ on $\bar{\Delta}(\lambda)$ satisfies*

$$\langle \tau_w 1_\lambda \otimes (v_1 \otimes \cdots \otimes v_l), \tau_{w'} 1_\lambda \otimes (v_1' \otimes \cdots \otimes v_l') \rangle = \delta_{w^{-1}w', x} \langle v_{y(1)}, v_1' \rangle_1 \cdots \langle v_{y(l)}, v_l' \rangle_l$$

for all $w, w' \in D_\lambda$ and $v_i, v_i' \in L(\lambda_i)$, where $\langle \cdot, \cdot \rangle_i$ is some choice of (non-degenerate) contravariant form on each $L(\lambda_i)$.

Proof. Let $L' := L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l)$ for short, and denote the product of the forms $\langle \cdot, \cdot \rangle_i$ for $i = 1, \dots, l$ by $\langle \cdot, \cdot \rangle'$, which is a non-degenerate form on L'. Recall from the proof of Lemma 2.13 that the contravariant form on $\bar{\Delta}(\lambda)$ is defined from $\langle v, v' \rangle := f(v)(v')$ where $f : \bar{\Delta}(\lambda) \to \bar{\nabla}(\lambda)$ is a non-zero homomorphism. We can identify $\bar{\nabla}(\lambda) = \mathrm{Hom}_{\mathbb{K}}(\bar{\Delta}(\lambda), \mathbb{K})$ with the coinduced module $q^{-s_\lambda} \mathrm{Hom}_{H_\lambda}(1_\lambda H_\alpha, L')$ so that $\theta : 1_\lambda H_\alpha \to L'$ is identified with the functional $\bar{\Delta}(\lambda) \mapsto \mathbb{K}, h 1_\lambda \otimes v \mapsto \langle \theta(1_\lambda T(h)), v \rangle'$. Then by adjointness of restriction and coinduction we get a canonical isomorphism

$$\mathrm{Hom}_{H_\alpha}(\bar{\Delta}(\lambda), \bar{\nabla}(\lambda)) \cong \mathrm{Hom}_{H_\lambda}(\mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda), q^{-s_\lambda} L').$$

Now we observe as in the proof of Lemma 2.12 that the top non-zero layer in the Mackey filtration of $\mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda)$ is isomorphic to $q^{-s_\lambda} L'$. In this way, we obtain an explicit homomorphism $\bar{f} : \mathrm{res}^\alpha_\lambda \bar{\Delta}(\lambda) \to q^{-s_\lambda} L'$ such that

$$\bar{f}(1_\lambda \tau_w 1_\lambda \otimes (v_1 \otimes \cdots \otimes v_l)) = \delta_{w,x} v_{y(1)} \otimes \cdots \otimes v_{y(l)}$$

for $w \in D_\lambda$ and $v_i \in L(\lambda_i)$. Then choose $f : \bar{\Delta}(\lambda) \to \bar{\nabla}(\lambda)$ so that it corresponds to $\bar{f}$ under the above isomorphism. This means that

$$\langle h 1_\lambda \otimes v, h' 1_\lambda \otimes v' \rangle = f(h 1_\lambda \otimes v)(h' 1_\lambda \otimes v') = \langle \bar{f}(1_\lambda T(h') h 1_\lambda \otimes v), v' \rangle'.$$

The lemma follows from the last two displayed formulae. □

Using this we can prove that the contravariant form is symmetric.

Lemma 2.15. *For each $\lambda \in \mathrm{KP}(\alpha)$ the form $\langle \cdot, \cdot \rangle$ on $\bar{\Delta}(\lambda)$ is symmetric.*

Proof. We proceed by induction on $\mathrm{ht}(\alpha)$. The base case $\alpha = \alpha_i$ is trivial as $L(\alpha_i)$ is one-dimensional. For the induction step, Lemma 2.14 reduces us to the case that $\lambda = (\alpha)$ for some non-simple $\alpha \in R^+$. Pick $i \in I$ and $m \geq 1$ such that $\mathrm{res}^\alpha_{\alpha - m\alpha_i, m\alpha_i} L(\alpha) \neq 0$, and either $\alpha - (m+1)\alpha_i \notin Q^+$ or $\mathrm{res}^\alpha_{\alpha-(m+1)\alpha_i,(m+1)\alpha_i} L(\alpha) = 0$. By general theory, $\mathrm{res}^\alpha_{\alpha - m\alpha_i, m\alpha_i} L(\alpha) \cong L(\mu) \boxtimes L(\alpha_i^m)$ for some $\mu \in \mathrm{KP}(\alpha - m\alpha_i)$; see e.g. [KL1, Lemma 3.7]. By Lemma 2.13, the restriction of the form $\langle \cdot, \cdot \rangle$ to this copy of $L(\mu) \boxtimes L(\alpha_i^m)$ is a non-degenerate contravariant form in the obvious sense, hence it is the product of contravariant forms on $L(\mu)$ and $L(\alpha_i^m)$. Both of these are already known to be symmetric by induction. This shows the form is symmetric on restriction to some non-zero word space of $L(\alpha)$. Since $L(\alpha)$ is irreducible, this implies it is symmetric on the entire module. □

Now we recall a conjecture from [KR2, Conjecture 7.3] asserting in finite type that the formal character of an irreducible H_α-module $L(\lambda)$ does not depend on the characteristic p of the ground field $\mathbb{K}$. Using geometric techniques, Williamson [Wi] has recently shown that this is false, and the question of finding a satisfactory bound on p remains open. The smallest counterexample found by Williamson (based ultimately on the counterexample from [KS]) is as follows.

Example 2.16. Assume we are in type A_5. Index the simple roots in the usual way 1—2—3—4—5 and choose the signs $\varepsilon_{i,j}$ in the definition of the KLR algebra so that $\varepsilon_{1,2} = \varepsilon_{2,3} = \varepsilon_{3,4} = \varepsilon_{4,5} = +$. The positive roots are $\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i \leq j \leq 5$. Let $\prec$ be the convex ordering defined so that $\alpha_{i,j} \prec \alpha_{k,l}$ if either $i < k$, or $i = k$ and $j < l$; cf. Example A.1 below. For this choice the module $L(\alpha_{i,j})$ is trivial to construct explicitly: it is the one-dimensional module spanned by a degree zero vector belonging to the $i(i+1)\cdots j$-word space. Let

$$\lambda := (\alpha_{4,5}, \alpha_{4,5}, \alpha_3, \alpha_3, \alpha_{2,4}, \alpha_{2,4}, \alpha_{1,2}, \alpha_{1,2}),$$
$$\boldsymbol{i} := 4534234523123412.$$

We claim that $1_{\boldsymbol{i}} L(\lambda)_0$ has dimension 2 if $\operatorname{char} \mathbb{K} = 2$ and dimension 3 in all other characteristics. To see this we compute the rank of the contravariant form on $1_{\boldsymbol{i}} \bar{\Delta}(\lambda)_0$. Let v span $L(\alpha_{4,5})^{\boxtimes 2} \boxtimes L(\alpha_3)^{\boxtimes 2} \boxtimes L(\alpha_{2,4})^{\boxtimes 2} \boxtimes L(\alpha_{1,2})^{\boxtimes 2}$. Adopting all the notation from Lemma 2.14, the vectors $\{\tau_w 1_\lambda \otimes v \,|\, w \in D_\lambda\}$ give a basis for $\bar{\Delta}(\lambda)$ with $1_\lambda \otimes v$ of degree 4 and $\tau_x 1_\lambda \otimes v$ of degree -4. We normalize the contravariant form on $\bar{\Delta}(\lambda)$ so that $\langle 1_\lambda \otimes v, \tau_x 1_\lambda \otimes v\rangle = -1$. Let

$$\begin{aligned}
a &:= \tau_3\tau_7\tau_6\tau_5\tau_4\tau_9\tau_8\tau_7\tau_6\tau_{12}\tau_{11}\tau_{13}\tau_{12},\\
b &:= \tau_3\tau_7\tau_6\tau_5\tau_4\tau_{12}\tau_{11}\tau_{10}\tau_9\tau_8\tau_7\tau_6\tau_{13}\tau_{12},\\
c_1 &:= \tau_2\tau_1\tau_3\tau_2,\\
c_2 &:= \tau_5,\\
c_3 &:= \tau_9\tau_8\tau_7\tau_{10}\tau_9\tau_8\tau_{11}\tau_{10}\tau_9,\\
c_4 &:= \tau_{14}\tau_{13}\tau_{15}\tau_{14}.
\end{aligned}$$

We have that $c_1c_2c_3c_4 1_\lambda \otimes v = \tau_x 1_\lambda \otimes v$ (as should be clear on drawing the appropriate diagrams), and $1_{\boldsymbol{i}} \bar{\Delta}(\lambda)_0$ is 5-dimensional with basis

$$ac_1c_2c_3c_4 1_\lambda \otimes v, bc_2c_3c_4 1_\lambda \otimes v, bc_1c_3c_4 1_\lambda \otimes v, bc_1c_2c_4 1_\lambda \otimes v, bc_1c_2c_3 1_\lambda \otimes v.$$

Using Lemma 2.14 and making some explicit but lengthy straightening calculations, one can then check that the Gram matrix of the contravariant form on $1_{\boldsymbol{i}} \bar{\Delta}(\lambda)_0$ with respect to this basis is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1\\ 1 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 1\\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

It remains to compute the rank of this matrix.

3. Standard modules

We continue to work with a fixed choice of convex ordering $\prec$ on R^+. In this section we will give an elementary definition first of *root modules* $\Delta(\alpha)$ categorifying the root elements r_α, then of *standard modules* $\Delta(\lambda)$ categorifying the PBW basis elements r_λ. We show that standard modules satisfy homological properties analogous to the standard modules of a quasi-hereditary algebra, hence in simply-laced types they are isomorphic to the modules $\widetilde{E}_b$ constructed using Saito reflection functors in [Ka, §4].

3.1. **Root modules.** We fix a positive root α throughout the subsection. All the good homological properties of KLR algebras proved in this article stem from the following key observation made in [M, §4].

Theorem 3.1. *For $d \geq 2$, we have that*

$$\mathrm{Ext}^1_{H_\alpha}(L(\alpha), L(\alpha)) \cong q_\alpha^{-2}\mathbb{K}, \qquad \mathrm{Ext}^d_{H_\alpha}(L(\alpha), L(\alpha)) = 0.$$

Proof. As explained in the second half of the proof of [M, Proposition 4.5], this is a consequence of the finiteness of the global dimension of H_α. The latter property is established in simply-laced types for $\mathbb{K}$ of characteristic zero in [Ka, Corollary 2.9] and in non-simply-laced types for $\mathbb{K}$ of arbitrary characteristic in [M, Theorem 4.6]. This leaves us with simply-laced types in positive characteristic. These can be treated by the same argument used in the non-simply-laced case in [M]. However some additional computations are needed which we postpone to the appendix; see Corollary A.11. □

Our first application is to the construction of root modules.

Lemma 3.2. *For $n \geq 0$, there exist unique (up to isomorphism) indecomposable H_α-modules $\Delta_n(\alpha)$ with $\Delta_0(\alpha) = 0$ such that there are short exact sequences*

$$0 \longrightarrow q_\alpha^{2(n-1)} L(\alpha) \xrightarrow{i_n} \Delta_n(\alpha) \xrightarrow{p_n} \Delta_{n-1}(\alpha) \longrightarrow 0, \tag{3.1}$$

$$0 \longrightarrow q_\alpha^{2} \Delta_{n-1}(\alpha) \xrightarrow{j_n} \Delta_n(\alpha) \xrightarrow{q_n} L(\alpha) \longrightarrow 0. \tag{3.2}$$

Moreover the following hold for all $n \geq 1$:

(1) $[\Delta_n(\alpha)] = \dfrac{1-q_\alpha^{2n}}{1-q_\alpha^2}[L(\alpha)]$;
(2) *$\Delta_n(\alpha)$ has head isomorphic to $L(\alpha)$ and socle isomorphic to $q_\alpha^{2(n-1)}L(\alpha)$;*
(3) *the map $i_n^* : \mathrm{Ext}^1_{H_\alpha}(\Delta_n(\alpha), L(\alpha)) \to \mathrm{Ext}^1_{H_\alpha}(q_\alpha^{2(n-1)}L(\alpha), L(\alpha))$ induced by i_n is an isomorphism, hence $\mathrm{Ext}^1_{H_\alpha}(\Delta_n(\alpha), L(\alpha)) \cong q_\alpha^{-2n}\mathbb{K}$;*
(4) $\mathrm{Ext}^d_{H_\alpha}(\Delta_n(\alpha), L(\alpha)) = 0$ *for all $d \geq 2$.*

Proof. Let $\Delta_0(\alpha) := 0$ and $\Delta_1(\alpha) := L(\alpha)$. The properties (1)–(4) hold when $n = 1$ by Theorem 3.1. Now suppose that $n \geq 2$ and that we have constructed $\Delta_{n-1}(\alpha)$ satisfying the properties (1)–(4). In particular by (3) we have that $\mathrm{Ext}^1_{H_\alpha}(\Delta_{n-1}(\alpha), L(\alpha)) \cong q_\alpha^{-2(n-1)}\mathbb{K}$, so there exists a unique (up to isomorphism) module $\Delta_n(\alpha)$ fitting into a non-split short exact sequence of the form (3.1). We must prove that $\Delta_n(\alpha)$ also satisfies the properties (1)–(4) (hence by (2) it is indecomposable) and that there exists a short exact sequence of the form (3.2). Property (1) is immediate from (3.1) and the induction hypothesis.

Applying $\mathrm{Hom}_{H_\alpha}(-, L(\alpha))$ to (3.1) using Theorem 3.1 and the induction hypothesis, we deduce that (4) holds. Moreover there is an exact sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathrm{Hom}_{H_\alpha}(\Delta_n(\alpha), L(\alpha)) \xrightarrow{f} q_\alpha^{-2(n-1)}\mathbb{K} \longrightarrow q_\alpha^{-2(n-1)}\mathbb{K}$$
$$\longrightarrow \mathrm{Ext}^1_{H_\alpha}(\Delta_n(\alpha), L(\alpha)) \xrightarrow{i_n^*} \mathrm{Ext}^1_{H_\alpha}(q_\alpha^{2(n-1)}L(\alpha), L(\alpha)) \longrightarrow 0.$$

The map f is zero, for otherwise there exists a non-zero homogeneous homomorphism $\Delta_n(\alpha) \to q_\alpha^{2(n-1)} L(\alpha)$, which is a contradiction as (3.1) is non-split. Hence $\mathrm{Hom}_{H_\alpha}(\Delta_n(\alpha), L(\alpha)) \cong \mathbb{K}$. Since all composition factors of $\Delta_n(\alpha)$ are of the form $L(\alpha)$ (up to shift) by (1), this shows $\Delta_n(\alpha)$ has irreducible head $L(\alpha)$. Then we see that ι_n^* is an isomorphism proving the first half of (3); the second half of (3) is immediate from Theorem 3.1.

To complete the proof of (2), it remains to compute the socle of $\Delta_n(\alpha)$. The image of the non-trivial extension represented by (3.1) under the map i_{n-1}^* is the extension represented by

$$0 \longrightarrow q_\alpha^{2(n-1)} L(\alpha) \xrightarrow{i_n} V \xrightarrow{p_n} q_\alpha^{2(n-2)} L(\alpha) \longrightarrow 0$$

where $V := p_n^{-1}(\mathrm{im}\, i_{n-1})$. Since i_{n-1}^* is an isomorphism by the induction hypothesis, this extension is non-trivial, hence $q_\alpha^{2(n-2)} L(\alpha)$ does not appear in the socle of V. If the socle of $\Delta_n(\alpha)$ is not irreducible then, in view of (3.2) and the induction hypothesis, its socle must be isomorphic to $q_\alpha^{2(n-1)} L(\alpha) \oplus q_\alpha^{2(n-2)} L(\alpha)$, contradicting the previous sentence. Hence $\mathrm{soc}\, \Delta_n(\alpha) \cong q_\alpha^{2(n-1)} L(\alpha)$.

Finally we prove the existence of the short exact sequence (3.2). This is just the same as (3.1) if $n = 2$, so assume $n \geq 3$. Applying $\mathrm{hom}_{H_\alpha}(q_\alpha^2 \Delta_{n-1}(\alpha), -)$ to (3.1) and using induction, we deduce that $\mathrm{hom}_{H_\alpha}(q_\alpha^2 \Delta_{n-1}(\alpha), \Delta_n(\alpha)) \cong \mathbb{K}$. Let $j_n : q_\alpha^2 \Delta_{n-1}(\alpha) \to \Delta_n(\alpha)$ be any non-zero homogeneous homomorphism. It must be injective since it is injective on $\mathrm{soc}\, q_\alpha^2 \Delta_n(\alpha)$ thanks to (1)–(2). Moreover $\mathrm{coker}\, j_n \cong L(\alpha)$ by (1). $\square$

This shows that there is an inverse system $\Delta_0(\alpha) \overset{p_1}{\twoheadleftarrow} \Delta_1(\alpha) \overset{p_2}{\twoheadleftarrow} \Delta_2(\alpha) \overset{p_3}{\twoheadleftarrow} \cdots$. Define the *root module*

$$\Delta(\alpha) := \varprojlim \Delta_n(\alpha).$$

Recalling (2.9), part (1) of the following theorem shows that $\Delta(\alpha)$ categorifies the root vector r_α.

Theorem 3.3. *There is a short exact sequence*

$$0 \longrightarrow q_\alpha^2 \Delta(\alpha) \xrightarrow{j} \Delta(\alpha) \longrightarrow L(\alpha) \longrightarrow 0. \tag{3.3}$$

Moreover:

(1) $\Delta(\alpha)$ *is a cyclic module with* $[\Delta(\alpha)] = [L(\alpha)]/(1 - q_\alpha^2)$*;*
(2) $\Delta(\alpha)$ *has irreducible head isomorphic to* $L(\alpha)$*;*
(3) *we have that* $\mathrm{Ext}^d_{H_\alpha}(\Delta(\alpha), V) = 0$ *for* $d \geq 1$ *and any finitely generated* H_α*-module* V *with all irreducible subquotients isomorphic to* $L(\alpha)$ *(up to degree shift);*
(4) $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) \cong \mathbb{K}[x]$ *for* x *in degree* $2d_\alpha$.

Proof. By definition of inverse limit, there are canonical maps $\pi_n : \Delta(\alpha) \twoheadrightarrow \Delta_n(\alpha)$ such that $p_n \circ \pi_n = \pi_{n-1}$ for each $n \geq 1$. For $i \in \mathbb{Z}$ the dimension of the graded component $\Delta_n(\alpha)_i$ is bounded independent of n, so the inverse system of vector spaces $\Delta_0(\alpha)_i \twoheadleftarrow \Delta_1(\alpha)_i \twoheadleftarrow \cdots$ stabilizes after finitely many terms. Since $\Delta(\alpha)_i = \varprojlim \Delta_n(\alpha)_i$, we deduce that $\bigcap_{n \geq 0} \ker \pi_n = 0$. Combined also with Lemma 3.2(1), it follows that $[\Delta(\alpha)] = [L(\alpha)]/(1 - q_\alpha^2)$. Also $\Delta(\alpha)$ is cyclic because each $\Delta_n(\alpha)$ is cyclic by Lemma 3.2(2). This proves (1).

In view of (1), the head of $\Delta(\alpha)$ is isomorphic to a finite direct sum of copies of $q_\alpha^{2n} L(\alpha)$ for $n \geq 0$. To deduce (2) we must show $\mathrm{hom}_{H_\alpha}(\Delta(\alpha), L(\alpha)) \cong \mathbb{K}$ and

$\hom_{H_\alpha}(\Delta(\alpha), q_\alpha^{2n}L(\alpha)) = 0$ for $n > 0$. Suppose we are given a non-zero homogeneous homomorphism $f : \Delta(\alpha) \to q_\alpha^{2n}L(\alpha)$ for some $n \geq 0$. By (1), all irreducible subquotients of $\ker \pi_{n+1}$ are of the form $q_\alpha^{2m}L(\alpha)$ for $m > n$, hence f factors through the quotient to induce $\bar{f} : \Delta_{n+1}(\alpha) \to q_\alpha^{2n}L(\alpha)$. It remains to apply Lemma 3.2(2).

Next we construct the short exact sequence (3.3). We claim that we can choose the injective homomorphisms j_n in (3.2) so that the following diagrams commute for all $n \geq 1$:

$$\begin{tikzcd} q_\alpha^2\Delta_{n-1}(\alpha) \arrow[d, "j_n"'] & q_\alpha^2\Delta_n(\alpha) \arrow[l, "p_n"'] \arrow[d, "j_{n+1}"] \\ \Delta_n(\alpha) & \Delta_{n+1}(\alpha). \arrow[l, "p_{n+1}"'] \end{tikzcd}$$

This is automatic if $n = 1$ (both compositions are zero), so assume $n \geq 2$ and that we are given j_n. The map $j_n \circ p_n$ is obviously non-zero, as is $p_{n+1} \circ j_{n+1}$ because $\operatorname{soc} \Delta_n(\alpha) \cong q_\alpha^{2(n-1)}L(\alpha)$. Also by Lemma 3.2(1)–(2), $\hom_{H_\alpha}(q_\alpha^2\Delta_n(\alpha), \Delta_n(\alpha))$ is one-dimensional. Hence j_{n+1} can be rescaled by a non-zero scalar if necessary to ensure that the above diagram commutes. This proves the claim. Hence we get induced an injective homogeneous homomorphism $j : q_\alpha^2\Delta(\alpha) \to \Delta(\alpha)$ such that $\pi_n \circ j = j_n \circ \pi_{n-1}$ for each $n \geq 1$. By (1) we have that $\operatorname{coker} j \cong L(\alpha)$ and (3.3) follows.

To prove (3), let V be as in its statement and take some fixed $d \geq 1$. By Theorem 3.1 and Lemma 1.1, we have that $\operatorname{Ext}^{d+1}_{H_\alpha}(L(\alpha), V) = 0$. Applying $\operatorname{Hom}_{H_\alpha}(-, V)$ to the short exact sequence (3.3), we deduce the existence of a homogeneous surjection

$$\operatorname{Ext}^d_{H_\alpha}(\Delta(\alpha), V) \twoheadrightarrow q_\alpha^{-2}\operatorname{Ext}^d_{H_\alpha}(\Delta(\alpha), V).$$

As V and $\Delta(\alpha)$ are both finitely generated, the graded vector space $\operatorname{Ext}^d_{H_\alpha}(\Delta(\alpha), V)$ is bounded below. The only way to avoid a contradiction is if $\operatorname{Ext}^d_{H_\alpha}(\Delta(\alpha), V)$ is zero.

Finally we prove (4). The map j in (3.3) can be viewed as an injective endomorphism $x \in \operatorname{End}_{H_\alpha}(\Delta(\alpha))_{2d_\alpha}$. It generates a free polynomial subalgebra $\mathbb{K}[x]$ of $\operatorname{End}_{H_\alpha}(\Delta(\alpha))$. But we also know that $\operatorname{Dim}\operatorname{End}_{H_\alpha}(\Delta(\alpha)) \leq \operatorname{Dim}\mathbb{K}[x]$ by (1)–(2). So we must have equality here. ◻

Corollary 3.4. *The functor* $\operatorname{Hom}_{H_\alpha}(\Delta(\alpha), -)$ *defines an equivalence from the category of finitely generated H_α-modules all of whose irreducible subquotients are isomorphic to $L(\alpha)$ (up to a shift) to the category of finitely generated* $\mathbb{K}[x]$*-modules (graded so that x is in degree $2d_\alpha$).*

Proof. The given category of H_α-modules is an abelian category. By Theorem 3.3(3), $\Delta(\alpha)$ is a projective generator. Hence the category is equivalent via the given functor to the category of finitely generated modules over $\operatorname{End}_{H_\alpha}(\Delta(\alpha))$, which is $\mathbb{K}[x]$ by Theorem 3.3(4). ◻

Corollary 3.5. *Any finitely generated H_α-module with all irreducible subquotients isomorphic to $L(\alpha)$ (up to a shift) is a finite direct sum of degree-shifted copies of the indecomposable modules $\Delta_n(\alpha)$* $(n \geq 1)$ *and* $\Delta(\alpha)$*.*

Proof. The algebra $\mathbb{K}[x]$ is a principal ideal domain. ◻

3.2. **Divided powers.** Throughout the subsection we fix $\alpha \in R^+$ of height n. We are going to compute the endomorphism algebra of $\Delta(\alpha)^{\circ m}$. Choose a non-zero homogeneous vector $\bar{v}_\alpha$ of minimal degree in $L(\alpha)$ and set $v_\alpha := 1 \otimes \bar{v}_\alpha \in \Delta(\alpha)$. This ensures that v_α is of minimal degree in $\Delta(\alpha)$ and that $\Delta(\alpha)$ is generated as an H_α-module by

v_α. Also pick a non-zero endomorphism $x \in \mathrm{End}_{H_\alpha}(\Delta(\alpha))_{2d_\alpha}$. By Theorem 3.3(4) we have that $\mathrm{End}_{H_\alpha}(\Delta(\alpha)) = \mathbb{K}[x]$, so that x is unique up to a scalar.

From the endomorphism x, we obtain commuting endomorphisms $x_1, \dots, x_m \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})_{2d_\alpha}$ with $x_i := \mathrm{id}^{\circ(i-1)} \circ x \circ \mathrm{id}^{\circ(m-i)}$. Similarly, the endomorphism τ from the following lemma yields $\tau_1, \dots, \tau_{m-1} \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})_{-2d_\alpha}$ with $\tau_i := \mathrm{id}^{\circ(i-1)} \circ \tau \circ \mathrm{id}^{\circ(m-i-1)}$. (It is a bit confusing here that we are using the same notation x_i and τ_i for these endomorphisms as we use for the elements of the KLR algebra, but hopefully it is clear from the context which we mean.)

Lemma 3.6. *Let $w \in S_{2n}$ be the permutation mapping $(1, \dots, n, n+1, \dots, 2n)$ to $(n+1, \dots, 2n, 1, \dots, n)$. There is a unique $H_{2\alpha}$-module homomorphism*

$$\tau : \Delta(\alpha) \circ \Delta(\alpha) \to \Delta(\alpha) \circ \Delta(\alpha)$$

of degree $-2d_\alpha$ such that $\tau(1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)) = \tau_w 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$.

Proof. We apply the Mackey theorem to $\mathrm{res}^{2\alpha}_{\alpha,\alpha} \Delta(\alpha) \circ \Delta(\alpha)$. By exactly the same argument as in the proof of Lemma 2.11, there are just two non-zero layers in the Mackey filtration, corresponding to the double coset representatives 1 and w. We deduce that there is a short exact sequence

$$0 \longrightarrow \Delta(\alpha) \boxtimes \Delta(\alpha) \xrightarrow{f} \mathrm{res}^{2\alpha}_{\alpha,\alpha} \Delta(\alpha) \circ \Delta(\alpha) \xrightarrow{g} q_\alpha^{-2} \Delta(\alpha) \boxtimes \Delta(\alpha) \longrightarrow 0$$

such that $f(v_\alpha \otimes v_\alpha) = 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ and $g(\tau_w 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)) = v_\alpha \otimes v_\alpha$. By Theorem 3.3(3), we have that

$$\mathrm{Ext}^1_{H_{\alpha,\alpha}}(\Delta(\alpha) \boxtimes \Delta(\alpha), \Delta(\alpha) \boxtimes \Delta(\alpha)) = 0.$$

So the short exact sequence splits. Let $\bar{g} : q_\alpha^{-2} \Delta(\alpha) \boxtimes \Delta(\alpha) \to \mathrm{res}^{2\alpha}_{\alpha,\alpha} \Delta(\alpha) \circ \Delta(\alpha)$ be the unique splitting. Since $\mathrm{im}\, f = 1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha))$ contains no non-zero vectors of degree $2\deg(v_\alpha) - 2d_\alpha$, we must have that $\bar{g}(v_\alpha \otimes v_\alpha) = \tau_w 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$. Applying Frobenius reciprocity, $\bar{g}$ induces a map τ as in the statement of the lemma. □

Lemma 3.7. *The endomorphisms $\tau_1, \dots, \tau_{m-1} \in \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$ square to zero and satisfy the usual type* A *braid relations.*

Proof. For the quadratic relation, we need to show in the setup of Lemma 3.6 that $\tau^2 = 0$. As a vector space, the Mackey theorem analysis from the proof of that lemma tells us that

$$1_{\alpha,\alpha}(\Delta(\alpha) \circ \Delta(\alpha)) = 1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha)) \oplus \tau_w 1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha)).$$

Thus the vector $\tau_w 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ is of minimal degree in $1_{\alpha,\alpha}(\Delta(\alpha) \circ \Delta(\alpha))$, namely, $2\deg(v_\alpha) - 2d_\alpha$. The vector $\tau_w^2 1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ is of strictly smaller degree $2\deg(v_\alpha) - 4d_\alpha$, hence it must be zero. This shows that τ^2 sends a generator of $\Delta(\alpha) \circ \Delta(\alpha)$ to zero, hence $\tau^2 = 0$.

For the braid relations, the commuting ones are trivial from the definitions. For the length three braid relation, it suffices to show that $\tau_1 \circ \tau_2 \circ \tau_1 = \tau_2 \circ \tau_1 \circ \tau_2$ working in $\mathrm{End}_{H_{3\alpha}}(\Delta(\alpha) \circ \Delta(\alpha) \circ \Delta(\alpha))$. Let $w_1, w_2 \in S_{3n}$ be the permutations mapping $(1, \dots, n, n+1, \dots, 2n, 2n+1, \dots, 3n)$ to $(n+1, \dots, 2n, 1, \dots, n, 2n+1, \dots, 3n)$ and $(1, \dots, n, 2n+1, \dots, 3n, n+1, \dots, 2n)$, respectively, and set $w_0 := w_1 w_2 w_1 = w_2 w_1 w_2$. By the defining relations for $H_{3\alpha}$, it is clear that $(\tau_{w_2} \tau_{w_1} \tau_{w_2} - \tau_{w_2} \tau_{w_1} \tau_{w_2}) 1_{\alpha,\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha \otimes v_\alpha)$ lies in

$$S := \sum_{w < w_0} \tau_w 1_{\alpha,\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha) \boxtimes \Delta(\alpha)).$$

By the Mackey theorem, we have that

$$S = \bigoplus_{w\in\{1,w_1,w_2,w_1w_2,w_2w_1\}} \tau_w 1_{\alpha,\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha) \boxtimes \Delta(\alpha)).$$

But the vector $(\tau_{w_2}\tau_{w_1}\tau_{w_2} - \tau_{w_2}\tau_{w_1}\tau_{w_2})1_{\alpha,\alpha,\alpha}\otimes(v_\alpha\otimes v_\alpha\otimes v_\alpha)$ is of degree $3\deg(v_\alpha)-6d_\alpha$, while all the vectors in S are of degree $\geq 3\deg(v_\alpha) - 4d_\alpha$. Hence this vector is zero, and we have shown that the endomorphisms $\tau_2 \circ \tau_1 \circ \tau_2$ and $\tau_1 \circ \tau_2 \circ \tau_1$ agree on the generator $1_{\alpha,\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha \otimes v_\alpha)$. Hence they are equal. □

In view of Lemma 3.7, we get well-defined endomorphisms τ_w of $\Delta(\alpha)^{\circ m}$ for each $w \in S_m$, defined as usual from any reduced expression for w. (This creates further ambiguity with the elements of the KLR algebra with the same name, but this is only temporary.)

Lemma 3.8. *The endomorphisms* $\{\tau_w \circ x_m^{k_m} \circ \cdots \circ x_1^{k_1} \mid w \in S_m, k_1, \dots, k_m \geq 0\}$ *give a basis for* $\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})$.

Proof. These endomorphisms are linearly independent because they produce linearly independent vectors when applied to $1_{\alpha,\dots,\alpha} \otimes (v_\alpha \otimes \cdots \otimes v_\alpha)$. It remains to show that

$$\mathrm{Dim}\,\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m}) \leq \sum_{w\in S_m} \frac{q_\alpha^{-2\ell(w)}}{(1-q_\alpha^2)^m}.$$

As $\Delta(\alpha)^{\boxtimes m}$ has head isomorphic to $L(\alpha)^{\boxtimes m}$ and $[\Delta(\alpha)^{\circ m}] = [L(\alpha)^{\circ m}]/(1-q_\alpha^2)^m$, we have by Frobenius reciprocity and Lemma 2.11 that

$$\begin{aligned}\mathrm{Dim}\,\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m}) &= \mathrm{Dim}\,\mathrm{Hom}_{H_{\alpha,\dots,\alpha}}(\Delta(\alpha)^{\boxtimes m}, \mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha}\Delta(\alpha)^{\circ m})\\ &\leq [\mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha}\Delta(\alpha)^{\circ m} : L(\alpha)^{\boxtimes m}]\\ &= [\mathrm{res}^{m\alpha}_{\alpha,\dots,\alpha}L(\alpha)^{\circ m} : L(\alpha)^{\boxtimes m}]/(1-q_\alpha^2)^m\\ &= q_\alpha^{-\frac{1}{2}m(m-1)}[m]^!_\alpha/(1-q_\alpha^2)^m.\end{aligned}$$

By the formula for the Poincaré polynomial of S_m this is $\sum_{w\in S_m} \frac{q_\alpha^{-2\ell(w)}}{(1-q_\alpha^2)^m}$. □

Lemma 3.9. *There is a* unique *choice for* $x \in \mathrm{End}_{H_\alpha}(\Delta(\alpha))_{2d_\alpha}$ *such that the following relations hold:* $\tau_i \circ x_j = x_j \circ \tau_i$ *for* $j \neq i, i+1$, $\tau_i \circ x_{i+1} = x_i \circ \tau_i + 1$ *and* $x_{i+1} \circ \tau_i = \tau_i \circ x_i + 1$.

Proof. The commuting relations are automatic. For the remaining relations, it suffices to show working in $\mathrm{End}_{H_{2\alpha}}(\Delta(\alpha) \circ \Delta(\alpha))$ that the (unique up to scalars) endomorphism $x \in \mathrm{End}_{H_\alpha}(\Delta(\alpha))_{2d_\alpha}$ can be chosen so that $\tau \circ x_2 = x_1 \circ \tau + 1$ and $x_2 \circ \tau = \tau \circ x_1 + 1$. Consider the endomorphisms

$$\theta_+ := \tau \circ x_2 - x_1 \circ \tau, \qquad \theta_- := \tau \circ x_1 - x_2 \circ \tau.$$

They are of degree zero and map $1_{\alpha,\alpha} \otimes (v_\alpha \otimes v_\alpha)$ into $1_{\alpha,\alpha} \otimes (\Delta(\alpha) \boxtimes \Delta(\alpha))$, so we deduce from Lemma 3.8 that $\theta_\pm = c_\pm$ for some scalars $c_\pm \in \mathbb{K}$. We have that

$$\tau x_1 x_2 - x_1 x_2 \tau = (x_2\tau + c_-)x_2 - x_2(\tau x_2 - c_+) = (c_- + c_+)x_2,$$

$$\tau x_1 x_2 - x_1 x_2 \tau = (x_1\tau + c_+)x_1 - x_1(\tau x_1 - c_-) = (c_+ + c_-)x_1.$$

Since x_1 and x_2 are linearly independent this implies that $c_- = -c_+$, and we have proved that $c_+ = -c_- = c$ for some $c \in \mathbb{K}$. It remains to show that $c \neq 0$, for then we can replace x by x/c and get that $\theta_+ = 1, \theta_- = -1$ as required. Suppose for a contradiction that $c = 0$. Then $\tau \circ x_1 = x_2 \circ \tau$ and $\tau \circ x_2 = x_1 \circ \tau$. This means that τ leaves invariant the submodule $S := \mathrm{im}\, x_1 + \mathrm{im}\, x_2$ of $\Delta(\alpha) \circ \Delta(\alpha)$, hence it induces a well-defined endomorphism $\bar{\tau}$ of the quotient $\Delta(\alpha) \circ \Delta(\alpha)/S$ with $\bar{\tau}^2 = 0$. But $\Delta(\alpha) \circ \Delta(\alpha)/S \cong L(\alpha) \circ L(\alpha)$, and under this isomorphism $\bar{\tau}$ corresponds to an endomorphism sending $1_{\alpha,\alpha} \otimes (\bar{v}_\alpha \otimes \bar{v}_\alpha)$ to $\tau_w 1_{\alpha,\alpha} \otimes (\bar{v}_\alpha \otimes \bar{v}_\alpha)$. This shows that $\mathrm{End}_{H_{2\alpha}}(L(\alpha) \circ L(\alpha))$ is more than one-dimensional, contradicting the irreducibility of this module from Theorem 2.8. □

Henceforth, we assume that the endomorphism x has been normalized according to Lemma 3.9. Recalling the definition of the nil Hecke algebra NH_m from §2.2, Lemmas 3.7–3.9 show that there is a unique algebra *isomorphism*

$$NH_m \xrightarrow{\sim} \mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}, \qquad x_i \mapsto x_i, \tau_j \mapsto \tau_j.$$

The op here means that we view $\Delta(\alpha)^{\circ m}$ as a *right* NH_m-module, i.e. it is an $(H_{m\alpha}, NH_m)$-bimodule (a convention which finally eliminates the confusion between the elements x_i, τ_j of $H_{m\alpha}$ and the elements of NH_m with the same name: they act on different sides). Finally define the *divided power module*

$$\Delta(\alpha^m) := q_\alpha^{\frac{1}{2}m(m-1)} \Delta(\alpha)^{\circ m} e_m \tag{3.4}$$

where $e_m \in NH_m$ is the idempotent (2.7).

Lemma 3.10. *We have that $\Delta(\alpha)^{\circ m} \cong [m]^!_\alpha \Delta(\alpha^m)$ as an $H_{m\alpha}$-module. Moreover $\Delta(\alpha^m)$ has irreducible head $L(\alpha^m)$, and in the Grothendieck group we have that $[\Delta(\alpha^m)] = [L(\alpha^m)]/(1 - q_\alpha^2)(1 - q_\alpha^4) \cdots (1 - q_\alpha^{2m})$.*

Proof. So far, we have identified the endomorphism algebra $\mathrm{End}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m})^{\mathrm{op}}$ with NH_m (graded so that x_i is in degree $2d_\alpha$ and τ_i is in degree $-2d_\alpha$). Since $NH_m \cong [m]^!_\alpha P$ where $P := q_\alpha^{\frac{1}{2}m(m-1)} NH_m e_m$, we deduce that

$$\Delta(\alpha)^{\circ m} = \Delta(\alpha)^{\circ m} \otimes_{NH_m} NH_m \cong [m]^!_\alpha \Delta(\alpha)^{\circ m} \otimes_{NH_m} P \cong [m]^!_\alpha \Delta(\alpha^m).$$

The fact that $[\Delta(\alpha^m)] = [L(\alpha^m)]/(1 - q_\alpha^2)(1 - q_\alpha^4) \cdots (1 - q_\alpha^{2m})$ follows from this using $[\Delta(\alpha)] = [L(\alpha)]/(1 - q_\alpha^2)$ and $L(\alpha^m) = q_\alpha^{\frac{1}{2}m(m-1)} L(\alpha)^{\circ m}$. Finally to show that the head of $\Delta(\alpha^m)$ is $L(\alpha^m)$, it suffices to show that

$$\mathrm{Dim}\,\mathrm{Hom}_{H_{m\alpha}}(\Delta(\alpha)^{\circ m}, L(\alpha^m)) = [m]^!_\alpha,$$

which follows from Theorem 3.3(2), Lemma 2.11 and Frobenius reciprocity. □

Thus we have constructed a module $\Delta(\alpha^m)$ which is equal in the Grothendieck group to the divided power $r_\alpha^m/[m]^!_\alpha$. More generally, for a Kostant partition $\lambda \in \mathrm{KP}$, gather together its equal parts to write it as $(\gamma_1^{m_1}, \dots, \gamma_s^{m_s})$ with $\gamma_1 \succ \cdots \succ \gamma_s$, then define the *standard module*

$$\Delta(\lambda) := \Delta(\gamma_1^{m_1}) \circ \cdots \circ \Delta(\gamma_s^{m_s}). \tag{3.5}$$

Recalling (2.10), the following theorem implies in particular that $[\Delta(\lambda)] = r_\lambda$.

Theorem 3.11. *For $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}$, we have that*

$$\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l) \cong [\lambda]^! \Delta(\lambda).$$

Moreover $V_0 := \Delta(\lambda)$ has an exhaustive filtration $V_0 \supset V_1 \supset V_2 \supset \cdots$ such that $V_0/V_1 \cong \bar{\Delta}(\lambda)$ and all other sections of the form $q^{2m}\bar{\Delta}(\lambda)$ for $m > 0$. Finally $\Delta(\lambda)$ has *irreducible head isomorphic to* $L(\lambda)$*, and*

$$[\Delta(\lambda)] = [\bar{\Delta}(\lambda)] \Big/ \prod_{\substack{\beta \in R^+ \\ 1 \leq r \leq m_\beta(\lambda)}} (1 - q_\beta^{2r}).$$

Proof. The isomorphism $\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l) \cong [\lambda]^! \Delta(\lambda)$ and the Grothendieck group identity both follow from Lemma 3.10. The existence of the filtration follows from Lemma 3.10 and exactness of induction. Finally, to show that $\Delta(\lambda)$ has irreducible head, the filtration together with Theorem 2.9 implies that the only module that could possibly appear with non-zero multiplicity in the head of $\Delta(\lambda)$ is $L(\lambda)$. Now calculate using Frobenius reciprocity, Lemma 2.12 and Lemma 3.10:

$$\begin{aligned}\operatorname{Dim}\operatorname{Hom}_{H_\alpha}(\Delta(\lambda), L(\lambda)) &= \operatorname{Dim}\operatorname{Hom}_{H_\alpha}(\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l), L(\lambda))/[\lambda]^! \\ &= \operatorname{Dim}\operatorname{Hom}_{H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_l}}(\Delta(\lambda_1) \boxtimes \cdots \boxtimes \Delta(\lambda_l), L(\lambda_1) \boxtimes \cdots \boxtimes L(\lambda_l)) = 1.\end{aligned}$$

□

3.3. **Standard homological properties.** In this subsection $\alpha \in Q^+$ is arbitrary. The following theorem extends the homological properties proved originally in [Ka, Theorem 4.12] to non-simply-laced types and to fields of positive characteristic. Recall $\bar{\nabla}(\lambda) = \bar{\Delta}(\lambda)^{\circledast}$.

Theorem 3.12. *Suppose that* $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}(\alpha)$*.*

(1) *We have that* $\operatorname{Ext}^d_{H_\alpha}(\Delta(\lambda), V) = 0$ *for all* $d \geq 1$ *and any finitely generated* H_α*-module* V *all of whose irreducible subquotients are of the form* $q^n L(\mu)$ *for* $n \in \mathbb{Z}$ *and* $\mu \in \mathrm{KP}(\alpha)$ *with* $\mu \not\succ \lambda$*.*

(2) *We have that*

$$\operatorname{Dim}\operatorname{Ext}^d_{H_\alpha}(\Delta(\lambda), \bar{\nabla}(\mu)) = \begin{cases} 1 & \text{if } d = 0 \text{ and } \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$$

for all $d \geq 0$ *and* $\mu \in \mathrm{KP}(\alpha)$*.*

Proof. (1) We first prove this in the special case that $V = \bar{\Delta}(\mu)$ for $\mu \not\succ \lambda$. By Theorem 3.11 we have that

$$\operatorname{Dim}\operatorname{Ext}^d_{H_\alpha}(\Delta(\lambda), \bar{\Delta}(\mu)) = \operatorname{Dim}\operatorname{Ext}^d_{H_\alpha}(\Delta(\lambda_1) \circ \cdots \circ \Delta(\lambda_l), \bar{\Delta}(\mu))/[\lambda]^!.$$

By generalized Frobenius reciprocity and Lemma 2.12, this is zero unless $\lambda \preceq \mu$. If $\lambda = \mu$ it equals

$$\sum_{d_1 + \cdots + d_l = d} \left(\prod_{k=1}^{l} \operatorname{Dim}\operatorname{Ext}^{d_k}_{H_{\lambda_k}}(\Delta(\lambda_k), L(\lambda_k)) \right),$$

which is zero by Theorem 3.3(3). Using this special case and arguing by induction on the ordering $\preceq$, it is straightforward to deduce that the result is also true if $V = L(\mu)$ for $\mu \not\succ \lambda$. Then the result for general V follows by Lemma 1.1.

(2) By dualizing Theorem 2.9, $\bar{\nabla}(\mu)$ has irreducible socle isomorphic to $L(\mu)$ and all its other composition are of the form $q^m L(\nu)$ for $\nu \prec \mu$ and $m \in \mathbb{Z}$. Now use Theorem 3.11(2) to deduce the result when $d = 0$. If $d \geq 1$ and $\mu \not\succ \lambda$ then we have that $\operatorname{Ext}^d_{H_\alpha}(\Delta(\lambda), \bar{\nabla}(\mu)) = 0$ by (1). Finally if $d \geq 1$ and $\mu \succ \lambda$, it suffices to show equivalently that $\operatorname{Ext}^d_{H_\alpha}(\bar{\Delta}(\mu), \nabla(\lambda)) = 0$ where $\nabla(\lambda) := \Delta(\lambda)^{\circledast}$. This follows from Lemma 2.12 and generalized Frobenius reciprocity once again. □

We say that an H_α-module V has a Δ-*flag* if there is a (finite!) filtration $V = V_0 \supset V_1 \supset \cdots \supset V_n = 0$ such that $V_i/V_{i-1} \cong q^{m_i}\Delta(\lambda_i)$ for each $i = 1, \dots, n$ and some $m_i \in \mathbb{Z}$, $\lambda_i \in \mathrm{KP}(\alpha)$. Note then by Theorem 3.12(2) that

$$[V : \Delta(\lambda)] := \sum_{\substack{1 \le i \le n \\ \lambda_i = \lambda}} q^{m_i} = \overline{\mathrm{Dim}\,\mathrm{Hom}_{H_\alpha}(V, \bar\nabla(\lambda))}$$

for each $\lambda \in \mathrm{KP}(\alpha)$, so that this multiplicity is well-defined independent of the particular choice of Δ-flag. The following theorem (and its proof) is analogous to a well-known result (and proof) in the context of quasi-hereditary algebras; see e.g. [Do, Proposition A2.2(iii)].

Theorem 3.13. *For $\alpha \in Q^+$, suppose that V is a finitely generated H_α-module with $\mathrm{Ext}^1_{H_\alpha}(V, \bar\nabla(\mu)) = 0$ for all $\mu \in \mathrm{KP}(\alpha)$. Then V has a Δ-flag.*

Proof. Let $\ell(V) \in \mathbb{N}$ denote the sum of the dimensions of $\mathrm{Hom}_{H_\alpha}(V, \bar\nabla(\lambda))$ for all $\lambda \in \mathrm{KP}(\alpha)$; this makes sense because V is finitely generated. We proceed by induction on $\ell(V)$, the result being trivial if $\ell(V) = 0$. Suppose that $\ell(V) > 0$. Let λ be minimal such that $\mathrm{Hom}_{H_\alpha}(V, L(\lambda)) \neq 0$. Then let $m \in \mathbb{Z}$ be minimal such that $\mathrm{hom}_{H_\alpha}(q^m V, L(\lambda)) \neq 0$.

We show in this paragraph that $\mathrm{Ext}^1_{H_\alpha}(V, L(\mu)) = 0$ for all $\mu \leq \lambda$. There is a short exact sequence $0 \to L(\mu) \to \bar\nabla(\mu) \to Q \to 0$ where all composition factors of Q are of the form $q^n L(\nu)$ for $n \in \mathbb{Z}$ and $\nu \prec \mu \leq \lambda$. By the minimality of λ, $\mathrm{Hom}_{H_\alpha}(V, Q) = 0$. Hence applying $\mathrm{Hom}_{H_\alpha}(V, -)$ to this short exact sequence, we obtain an exact sequence $0 \to \mathrm{Ext}^1_{H_\alpha}(V, L(\mu)) \to \mathrm{Ext}^1_{H_\alpha}(V, \bar\nabla(\mu)) = 0$. We are done.

Now recall by Theorem 3.11 that $\Delta(\lambda)$ has irreducible head isomorphic to $L(\lambda)$. In this paragraph, we show that there is a homogeneous surjection $q^m V \twoheadrightarrow \Delta(\lambda)$ by showing that the natural map $\mathrm{hom}_{H_\alpha}(q^m V, \Delta(\lambda)) \to \mathrm{hom}_{H_\alpha}(q^m V, L(\lambda))$ is surjective. The long exact sequence obtained by applying $\mathrm{hom}_{H_\alpha}(q^m V, -)$ to the short exact sequence $0 \to \mathrm{rad}\,\Delta(\lambda) \to \Delta(\lambda) \to L(\lambda) \to 0$ gives us an exact sequence

$$\mathrm{hom}_{H_\alpha}(q^m V, \Delta(\lambda)) \to \mathrm{hom}_{H_\alpha}(q^m V, L(\lambda)) \to \mathrm{ext}^1_{H_\alpha}(q^m V, \mathrm{rad}\,\Delta(\lambda)).$$

Thus we are reduced to showing that $\mathrm{ext}^1_{H_\alpha}(q^m V, \mathrm{rad}\,\Delta(\lambda)) = 0$. This follows using the previous paragraph and Lemma 1.1, noting by Theorems 2.9 and 3.11 that all irreducible subquotients of $\mathrm{rad}\,\Delta(\lambda)$ are of the form $q^n L(\mu)$ for $n \in \mathbb{Z}$ and $\mu \leq \lambda$.

We have now proved that there is a short exact sequence

$$0 \to U \to V \to q^{-m}\Delta(\lambda) \to 0$$

for some submodule U of V. Applying $\mathrm{Hom}_{H_\alpha}(-, \bar\nabla(\mu))$ we get from the long exact sequence and Theorem 3.12(2) that $\ell(U) < \ell(V)$ and $\mathrm{Ext}^1_{H_\alpha}(U, \bar\nabla(\mu)) = 0$ for all $\mu \in \mathrm{KP}(\alpha)$. Thus by induction U has a Δ-flag, hence so does V. $\square$

As a corollary we obtain "BGG reciprocity." For simply-laced types in characteristic zero, this was noted already in [Ka, Remark 4.17] (for convex orderings that are adapted to the orientation of the quiver).

Corollary 3.14. *For any $\alpha \in Q^+$ and $\lambda \in \mathrm{KP}(\alpha)$, the projective module $P(\lambda)$ has a Δ-flag with $[P(\lambda) : \Delta(\mu)] = [\bar\Delta(\mu) : L(\lambda)]$ (the latter notation denotes graded Jordan-Hölder multiplicity).*

Proof. The theorem immediately implies that $P(\lambda)$ has a Δ-flag. Moreover

$$[P(\lambda):\Delta(\mu)] = \overline{\operatorname{Dim}\operatorname{Hom}_{H_\alpha}[P(\lambda),\bar{\bar\nabla}(\mu)]} = \overline{[\bar{\bar\nabla}(\mu):L(\lambda)]} = [\bar\Delta(\mu):L(\lambda)]$$

as $L(\lambda)^{\circledast} \cong L(\lambda)$. □

Corollary 3.15. *For any $\alpha \in Q^+$, we have that*

$$\operatorname{Dim} H_\alpha = \sum_{\lambda\in\mathrm{KP}(\alpha)} (\operatorname{Dim}\Delta(\lambda))(\operatorname{Dim}\bar\Delta(\lambda)) = \sum_{\lambda\in\mathrm{KP}(\alpha)} (\operatorname{Dim}\bar\Delta(\lambda))^2 \Big/ \prod_{\substack{\beta\in R^+\\ 1\le r\le m_\beta(\lambda)}} (1-q_\beta^{2r}).$$

Proof. Again H_α has a Δ-flag by the theorem, so its dimension is given by

$$\begin{aligned}\operatorname{Dim} H_\alpha &= \sum_{\lambda\in\mathrm{KP}(\alpha)} (\operatorname{Dim}\Delta(\lambda))(\overline{\operatorname{Dim}\operatorname{Hom}_{H_\alpha}(H_\alpha,\bar{\bar\nabla}(\lambda))})\\ &= \sum_{\lambda\in\mathrm{KP}(\alpha)} (\operatorname{Dim}\Delta(\lambda))(\overline{\operatorname{Dim}\bar{\bar\nabla}(\lambda)}) = \sum_{\lambda\in\mathrm{KP}(\alpha)} (\operatorname{Dim}\Delta(\lambda))(\operatorname{Dim}\bar\Delta(\lambda)).\end{aligned}$$

To deduce the second equality use the last part of Theorem 3.11. □

Corollary 3.16. *For any $\lambda \in \mathrm{KP}$, we have that*

$$\Delta(\lambda) \cong P(\lambda) \Big/ \sum_{\mu\not\leq\lambda}\ \sum_{f:P(\mu)\to P(\lambda)} \operatorname{im} f,$$

$$\bar\Delta(\lambda) \cong P(\lambda) \Big/ \sum_{\mu\not<\lambda}\ \sum_{f:P(\mu)\to \operatorname{rad} P(\lambda)} \operatorname{im} f,$$

summing over all (not necessarily homogeneous) homomorphisms f.

Proof. Any Δ-flag of $P(\lambda)$ must have $\Delta(\lambda)$ appearing at the top. Now apply Theorems 3.11 and 2.9. □

Remark 3.17. In simply-laced types with char $\mathbb{K} = 0$, Corollary 3.16 implies that our modules $\Delta(\lambda)$ and $\bar\Delta(\lambda)$ coincide with the modules $\widetilde{E}_b$ and E_b from [Ka, Corollary 4.18] (for $b \in B(\infty)$ chosen so that $L(\lambda) \cong L_b$).

4. Minimal pairs

In this section we show that the root modules $\Delta(\alpha)$ fit into some short exact sequences, giving an alternative inductive way to deduce their properties. We apply this to bound the projective dimension of standard modules, then construct some projective resolutions of root modules. The key to the proofs is a useful recursive formula for the root vectors r_α. This involves certain scale factors which were rather mysterious before; cf. [Le]. As usual we work with a fixed convex ordering $\prec$ on R^+.

4.1. **Scale factors.** As in [M], we refer to the pairs $\lambda = (\beta,\gamma)$ from the statement of Lemma 2.6 as the *minimal pairs* for $\alpha \in R^+$. Equivalently, a minimal pair for α is a pair (β,γ) of positive roots with $\beta+\gamma = \alpha$ and $\beta \succ \gamma$ such that there exists no other pair (β',γ') of positive roots with $\beta'+\gamma' = \alpha$ and $\beta \succ \beta' \succ \alpha \succ \gamma' \succ \gamma$. Let $\mathrm{MP}(\alpha)$ denote the set of all minimal pairs for α.

For $\lambda = (\beta,\gamma) \in \mathrm{MP}(\alpha)$, it is immediate from Theorem 2.9 and the minimality of λ that all composition factors of $\operatorname{rad}\bar\Delta(\lambda)$ are isomorphic to $L(\alpha)$ (up to degree shift).

Since $\bar{\Delta}(\lambda) = L(\beta) \circ L(\gamma)$ and $(L(\beta) \circ L(\gamma))^{\circledast} \cong q^{\beta\cdot\gamma} L(\gamma) \circ L(\beta)$ by Lemma 2.3, we deduce that there are short exact sequences

$$0 \longrightarrow q^{-\beta\cdot\gamma} M^{\circledast} \longrightarrow L(\beta) \circ L(\gamma) \longrightarrow L(\lambda) \longrightarrow 0, \tag{4.1}$$

$$0 \longrightarrow q^{-\beta\cdot\gamma} L(\lambda) \longrightarrow L(\gamma) \circ L(\beta) \longrightarrow M \longrightarrow 0, \tag{4.2}$$

where $M := q^{-\beta\cdot\gamma}(\mathrm{rad}\,\bar{\Delta}(\lambda))^{\circledast}$ is a finite dimensional module with all composition factors isomorphic to $L(\alpha)$ (up to degree shift). For $\beta, \gamma \in R$, let

$$p_{\beta,\gamma} := \max\,(p \in \mathbb{Z} \,|\, \beta - p\gamma \in R)\,.$$

Lemma 4.1. *For any $\alpha, \beta, \gamma \in R^+$ with $\beta + \gamma = \alpha$, we have that*

$$d_\alpha(p_{\beta,\gamma} - \beta\cdot\gamma) = d_\beta d_\gamma(p_{\beta,\gamma} + 1), \qquad [d_\alpha][p_{\beta,\gamma} - \beta\cdot\gamma] = [d_\beta][d_\gamma][p_{\beta,\gamma} + 1].$$

Proof. By inspection of the rank two root systems, we have that

$$p_{\beta,\gamma} = \begin{cases} 2 & \text{if } d_\alpha = 3 \text{ and } d_\beta = d_\gamma = 1, \\ 1 & \text{if } d_\alpha = 2 \text{ and } d_\beta = d_\gamma = 1, \\ 1 & \text{if } d_\alpha = d_\beta = d_\gamma = 1 \text{ in a subsystem of type } \mathrm{G}_2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Moreover in the last case we have that $d_\alpha = \min(d_\beta, d_\gamma)$. Now consider the four cases in turn, noting also that $\beta\cdot\gamma = d_\alpha - d_\beta - d_\gamma$. ◻

Theorem 4.2. *Let (β, γ) be a minimal pair for $\alpha \in R^+$. Then*

$$r_\gamma r_\beta - q^{-\beta\cdot\gamma} r_\beta r_\gamma = [p_{\beta,\gamma} + 1] r_\alpha, \tag{4.4}$$

$$r^*_\gamma r^*_\beta - q^{-\beta\cdot\gamma} r^*_\beta r^*_\gamma = q^{-p_{\beta,\gamma}}(1 - q^{2(p_{\beta,\gamma} - \beta\cdot\gamma)}) r^*_\alpha. \tag{4.5}$$

Proof. In this paragraph we consider the special case that β is simple. Under this assumption [KL1, Lemma 3.9] shows that $L(\gamma) \circ L(\beta)$ has irreducible head, and moreover no composition factors of $\mathrm{rad}\,(L(\gamma) \circ L(\beta))$ are isomorphic to its head (up to degree shift). We deduce that the module M in (4.2) is isomorphic to $q^{-p} L(\alpha)$ for some $p \in \mathbb{Z}$. Using this and considering the Grothendieck group identities obtained from (4.1)–(4.2), we deduce that

$$r^*_\gamma r^*_\beta - q^{-\beta\cdot\gamma} r^*_\beta r^*_\gamma = q^{-p}(1 - q^{2(p - \beta\cdot\gamma)}) r^*_\alpha. \tag{4.6}$$

Rearranging this using (2.9) and $\beta\cdot\gamma = d_\alpha - d_\beta - d_\gamma$, we get that

$$r_\gamma r_\beta - q^{-\beta\cdot\gamma} r_\beta r_\gamma = \frac{[d_\alpha][p - \beta\cdot\gamma]}{[d_\beta][d_\gamma]} r_\alpha. \tag{4.7}$$

Let us show further that $p \geq \beta\cdot\gamma$. Applying $(\cdot, r^*_\alpha)$ to (4.7), we deduce that

$$[d_\alpha][p - \beta\cdot\gamma] = [d_\beta][d_\gamma](r_\gamma r_\beta - q^{-\beta\cdot\gamma} r_\beta r_\gamma, r^*_\alpha).$$

Since $(r_\beta r_\gamma, r^*_\alpha) = (r_\beta \otimes r_\gamma, r(r^*_\alpha))$, where r is the twisted coproduct from (2.1) which corresponds to restriction under the categorification theorem, we see that $(r_\beta r_\gamma, r^*_\alpha)$ is the graded composition multiplicity $[\mathrm{res}^\alpha_{\beta,\gamma} L(\alpha), L(\beta) \boxtimes L(\gamma)]$, which is zero by Lemma 2.12. Similarly $(r_\gamma r_\beta, r^*_\alpha) = [\mathrm{res}^\alpha_{\gamma,\beta} L(\alpha) : L(\gamma) \boxtimes L(\beta)]$, which lies in $\mathbb{N}[q, q^{-1}]$. Altogether this shows that $[d_\alpha][p - \beta\cdot\gamma] \in \mathbb{N}[q, q^{-1}]$, hence indeed we must have that $p \geq \beta\cdot\gamma$. Finally, when we specialize at $q = 1$, the canonical basis elements $\{r_\alpha\}$ become Chevalley basis elements $\{e_\alpha\}$, and the identity (4.7) becomes

$$[e_\gamma, e_\beta] = \frac{d_\alpha(p - \beta\cdot\gamma)}{d_\beta d_\gamma} e_\alpha.$$

But for a Chevalley basis, $[e_\gamma, e_\beta] = \pm(p_{\beta,\gamma}+1)e_\alpha$. Since $p \geq \beta\cdot\gamma$, we deduce from Lemma 4.1 that $p = p_{\beta,\gamma}$. Hence (4.6) proves (4.5). Also using Lemma 4.1 once more (4.7) proves (4.4).

Now we deduce the general case. Note it is sufficient just to prove (4.4), for then (4.5) follows by rearranging using (2.9) and Lemma 4.1. Let $w_0 = s_{i_1}\cdots s_{i_N}$ be the reduced expression of w_0 corresponding to the given convex ordering $\prec$. Since $\gamma \prec \alpha \prec \beta$, there exist $c < a < b$ such that $\alpha = s_{i_1}\cdots s_{i_{a-1}}(\alpha_{i_a}), \beta = s_{i_1}\cdots s_{i_{b-1}}(\alpha_{i_b})$ and $\gamma = s_{i_1}\cdots s_{i_{c-1}}(\alpha_{i_c})$. Working now in $U_q(\mathfrak{g})$, we need to prove that

$$[p_{\beta,\gamma}+1]T_{i_1}\cdots T_{i_{a-1}}(E_{i_a}) = T_{i_1}\cdots T_{i_{c-1}}(E_{i_c})T_{i_1}\cdots T_{i_{b-1}}(E_{i_b}) \\ - q^{-\beta\cdot\gamma}T_{i_1}\cdots T_{i_{b-1}}(E_{i_b})T_{i_1}\cdots T_{i_{c-1}}(E_{i_c}).$$

Let $s_{j_1}\cdots s_{j_{N-b}}$ be a reduced expression for $w_0 s_{i_b}\cdots s_{i_1}$ and $\prec'$ be the convex ordering corresponding to the decomposition $w_0 = s_{j_1}\cdots s_{j_{N-b}}s_{i_1}\cdots s_{i_b}$. Let $\alpha' := s_{j_1}\cdots s_{j_{N-b}}(\alpha)$, $\beta' := s_{j_1}\cdots s_{j_{N-b}}(\beta)$ and $\gamma' := s_{j_1}\cdots s_{j_{N-b}}(\gamma)$. Note that β' is simple and (β',γ') is a minimal pair for α' with respect to the convex ordering $\prec'$. Acting with $T_{j_1}\cdots T_{j_{N-b}}$, the identity we are trying to prove is equivalent to the identity

$$[p_{\beta,\gamma}+1]T_{j_1}\cdots T_{j_{N-b}}T_{i_1}\cdots T_{i_{a-1}}(E_{i_a}) = \\ T_{j_1}\cdots T_{j_{N-b}}T_{i_1}\cdots T_{i_{c-1}}(E_{i_c})T_{j_1}\cdots T_{j_{N-b}}T_{i_1}\cdots T_{i_{b-1}}(E_{i_b}) \\ - q^{-\beta\cdot\gamma}T_{j_1}\cdots T_{j_{N-b}}T_{i_1}\cdots T_{i_{b-1}}(E_{i_b})T_{j_1}\cdots T_{j_{N-b}}T_{i_1}\cdots T_{i_{c-1}}(E_{i_c}).$$

But in $\mathbf{f}$ this is saying simply that $[p_{\beta',\gamma'}+1]r_{\alpha'} = r_{\gamma'}r_{\beta'} - q^{-\beta'\cdot\gamma'}r_{\beta'}r_{\gamma'}$, where the root elements here are defined with respect to the new convex ordering $\prec'$. This follows from the special case treated in the previous paragraph. □

Corollary 4.3. *Let (β,γ) be a minimal pair for $\alpha \in R^+$. In the Grothendieck group, we have that* $\left[\mathrm{res}^\alpha_{\gamma,\beta}L(\alpha)\right] = [p_{\beta,\gamma}+1][L(\gamma)\boxtimes L(\beta)]$.

Proof. By [M, Lemma 4.1], $\left[\mathrm{res}^\alpha_{\gamma,\beta}L(\alpha)\right]$ is a scalar multiple of $[L(\gamma)\boxtimes L(\beta)]$. To compute the scalar we make a computation with Lusztig's form like we did in the proof of Theorem 4.2:

$$(r_\gamma\otimes r_\beta, r(r^*_\alpha)) = (r_\gamma r_\beta, r^*_\alpha) = (r_\gamma r_\beta - q^{-\beta\cdot\gamma}r_\beta r_\gamma, r^*_\alpha) + q^{-\beta\cdot\gamma}(r_\beta r_\gamma, r^*_\alpha) \\ = [p_{\beta,\gamma}+1](r_\alpha, r^*_\alpha) + q(r_\beta\otimes r_\gamma, r(r^*_\alpha)) = [p_{\beta,\gamma}+1].$$

□

Remark 4.4. We will show in Theorem 4.7 below that the module M in (4.2) is isomorphic to $q^{-p_{\beta,\gamma}}L(\alpha)$. Hence for $(\beta,\gamma)\in \mathrm{MP}(\alpha)$ the module $L(\gamma)\circ L(\beta)$ has irreducible head $q^{-p_{\beta,\gamma}}L(\alpha)$. Applying Frobenius reciprocity, it follows that the self-dual module $\mathrm{res}^\alpha_{\gamma,\beta}L(\alpha)$ has irreducible socle $q^{p_{\beta,\gamma}}L(\alpha)$. Hence it is uniserial with composition factors as described by Corollary 4.3.

4.2. **Leclerc's algorithm.** Theorems 2.2 and 2.9 imply that

$$\mathsf{b}^*(r^*_\lambda) = r^*_\lambda + (\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } r^*_\mu \text{ for } \mu \prec \lambda). \tag{4.8}$$

Hence by duality we also have that

$$\mathsf{b}(r_\lambda) = r_\lambda + (\text{a } \mathbb{Z}[q,q^{-1}]\text{-linear combination of } r_\mu \text{ for } \mu \succ \lambda). \tag{4.9}$$

There are several other ways to prove the unitriangularity of the transition matrices here: the identity (4.8) follows from the b^*-invariance of the dual root vectors noted earlier together with the Levendorskii-Soibelman formula [LS, Proposition 5.5.2]; the identity (4.9) can be deduced directly starting from [L3, Proposition 1.9]. Combining (4.8)–(4.9) with Lusztig's lemma, it follows that there exist unique bases $\{b_\lambda \mid \lambda \in \mathrm{KP}\}$ and $\{b^*_\lambda \mid \lambda \in \mathrm{KP}\}$ for $\mathbf{f}_{\mathscr{A}}$ and $\mathbf{f}^*_{\mathscr{A}}$, respectively, such that

$$\mathsf{b}(b_\lambda) = b_\lambda, \qquad b_\lambda = r_\lambda + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } r_\mu \text{ for } \mu > \lambda), \tag{4.10}$$

$$\mathsf{b}^*(b^*_\lambda) = b^*_\lambda, \qquad b^*_\lambda = r^*_\lambda + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } r^*_\mu \text{ for } \mu < \lambda). \tag{4.11}$$

Of course these are the canonical and dual canonical bases for $\mathbf{f}$, respectively, as follows by the definition in [L1] in simply-laced types or [S] in non-simply-laced types.

For simply-laced types over fields of characteristic zero, the results of Rouquier [R2, §5] and Varagnolo-Vasserot [VV] show under the identification from Theorem 2.2 that $b_\lambda = [P(\lambda)]$, hence also $b^*_\lambda = [L(\lambda)]$, for each $\lambda \in \mathrm{KP}$. In all types, there is a recursive algorithm due to Leclerc [Le, §5.5] to compute b^*_λ, or rather its image $\mathrm{Ch}\, b^*_\lambda$ in the quantum shuffle algebra. Putting these two statements together, we obtain an effective algorithm to compute the characters of the irreducible H_α-modules in all simply-laced types over fields of characteristic zero. At present there is no reasonable way to compute the irreducible characters in non-simply-laced types or in positive characteristic; the approach via the contravariant form illustrated by Example 2.16 is seldom feasible in practice.

In the remainder of the subsection, we explain how to modify Leclerc's algorithm for computing dual canonical bases so that it works for an arbitrary convex ordering (rather than just for Lyndon orderings). Assume for this that we have chosen a minimal pair $\mathrm{mp}(\alpha) \in \mathrm{MP}(\alpha)$ for each $\alpha \in R^+$ of height at least two. Dependent on these choices, we recursively define a word $\boldsymbol{i}_\alpha \in \langle I\rangle_\alpha$ and a bar-invariant Laurent polynomial $\kappa_\alpha \in \mathscr{A}$: for $i \in I$ set $\boldsymbol{i}_{\alpha_i} := i$ and $\kappa_{\alpha_i} := 1$; then for $\alpha \in R^+$ of height ≥ 2 suppose that $(\beta, \gamma) = \mathrm{mp}(\alpha)$ and set

$$\boldsymbol{i}_\alpha := \boldsymbol{i}_\gamma \boldsymbol{i}_\beta, \qquad \kappa_\alpha := [p_{\beta,\gamma} + 1]\kappa_\beta\kappa_\gamma. \tag{4.12}$$

For example, in simply-laced types we have that $\kappa_\alpha = 1$ for all $\alpha \in R^+$; this is also the case in non-simply-laced types for *multiplicity-free* positive roots, i.e. roots $\alpha = \sum_{i\in I} c_i\alpha_i$ with $c_i \in \{0, 1\}$ for all i. Then for $\lambda = (\lambda_1, \dots, \lambda_l) \in \mathrm{KP}$ let

$$\boldsymbol{i}_\lambda := \boldsymbol{i}_{\lambda_1} \cdots \boldsymbol{i}_{\lambda_l}, \qquad \kappa_\lambda := [\lambda]^! \kappa_{\lambda_1} \cdots \kappa_{\lambda_l}. \tag{4.13}$$

Part (1) of the following lemma shows that the words $\boldsymbol{i}_\lambda$ distinguish irreducible modules, generalizing [KR2, Theorem 7.2(ii)].

Lemma 4.5. *The following hold for any $\alpha \in Q^+$ and $\lambda, \mu \in \mathrm{KP}(\alpha)$.*

(1) *We have that* $\mathrm{Dim}\, 1_{\boldsymbol{i}_\mu} L(\lambda) = 0$ *if $\mu \not\leq \lambda$, and* $\mathrm{Dim}\, 1_{\boldsymbol{i}_\lambda} L(\lambda) = \kappa_\lambda$.

(2) *The $\boldsymbol{i}_\mu$-coefficient of* $\mathrm{Ch}\, b^*_\lambda$ *is zero if $\mu \not\leq \lambda$, and the $\boldsymbol{i}_\lambda$-coefficient of* $\mathrm{Ch}\, b^*_\lambda$ *is equal to κ_λ.*

Proof. (1) Since $L(\lambda)$ is a quotient of $\bar{\Delta}(\lambda)$, the first statement is immediate from Lemma 2.12. Using also the triangularity property from Theorem 2.9, the same lemma reduces the proof of the second statement to showing that $\mathrm{Dim}\, 1_{\boldsymbol{i}_\alpha} L(\alpha) = \kappa_\alpha$ for $\alpha \in R^+$. To see this proceed by induction on height. For the induction step, suppose that $(\beta, \gamma) = \mathrm{mp}(\alpha)$ and apply Corollary 4.3:

$$\mathrm{Dim}\, 1_{\boldsymbol{i}_\alpha} L(\alpha) = [p_{\beta,\gamma} + 1](\mathrm{Dim}\, 1_{\boldsymbol{i}_\beta} L(\beta))(\mathrm{Dim}\, 1_{\boldsymbol{i}_\gamma} L(\gamma)) = [p_{\beta,\gamma} + 1]\kappa_\beta\kappa_\gamma = \kappa_\alpha.$$

(2) Repeat the proof of (1) working in $\mathbf{f}^*_{\mathscr{A}}$ rather than in $[\mathrm{Rep}(H)]$, using $[\bar{\Delta}(\lambda)] = r^*_\lambda$, $[L(\alpha)] = r^*_\alpha$ and the triangularity from (4.11). $\square$

Now we can explain the algorithm to compute $\mathrm{Ch}\, b^*_\lambda$ for $\lambda \in \mathrm{KP}$. This goes by induction on the partial order $\preceq$, so we assume $\mathrm{Ch}\, b^*_\mu$ is known for all $\mu \prec \lambda$. We can compute $\mathrm{Ch}\, r^*_\alpha \in \mathscr{A}\langle I\rangle$ for any $\alpha \in R^+$ by using the recursive formula obtained by applying Ch to the identity (4.5). Hence we can compute $\chi := \mathrm{Ch}\, r^*_\lambda \in \mathscr{A}\langle I\rangle$. Now inspect the $\boldsymbol{i}_\mu$-coefficients of χ for $\mu \prec \lambda$. If they are all bar-invariant then $\chi = \mathrm{Ch}\, b^*_\lambda$ and we are done. Otherwise, let $\mu \prec \lambda$ be maximal such that the coefficient $a(q)$ of $\boldsymbol{i}_\mu$ in χ is not bar-invariant; Lemma 4.5(2) and (4.11) imply that there is a unique $c(q) \in q\mathbb{Z}[q]$ such that $a(q) - c(q)\kappa_\mu$ is bar-invariant; then subtract $c(q)\,\mathrm{Ch}\, b^*_\mu$ from χ and repeat.

Example 4.6. The following example is mostly taken from [Le, §5.5.4]. Suppose we are in type G_2 with simple roots α_1 and α_2 chosen so that they are short and long, respectively. There are just two convex orderings. We consider the one in which $\alpha_1 \prec \alpha_2$. Then the above algorithm gives:

$\mathrm{Ch}\, b^*_{(\alpha_1)} = 1;$
$\mathrm{Ch}\, b^*_{(3\alpha_1+\alpha_2)} = [2]_1[3]_1 1112,$
$\mathrm{Ch}\, b^*_{(2\alpha_1+\alpha_2,\alpha_1)} = [2]_1 1121,$
$\mathrm{Ch}\, b^*_{(\alpha_1+\alpha_2,\alpha_1,\alpha_1)} = [2]_1 1211,$
$\mathrm{Ch}\, b^*_{(\alpha_2,\alpha_1,\alpha_1,\alpha_1)} = [2]_1[3]_1 2111;$
$\mathrm{Ch}\, b^*_{(2\alpha_1+\alpha_2)} = [2]_1 112,$
$\mathrm{Ch}\, b^*_{(\alpha_1+\alpha_2,\alpha_1)} = 121,$
$\mathrm{Ch}\, b^*_{(\alpha_2,\alpha_1,\alpha_1)} = [2]_1 211;$
$\mathrm{Ch}\, b^*_{(3\alpha_1+2\alpha_2)} = [2]_2[2]_1[3]_1 11122 + [2]_1[3]_1 11212,$
$\mathrm{Ch}\, b^*_{(\alpha_1+\alpha_2,2\alpha_1+\alpha_2)} = [2]_1 12112,$
$\mathrm{Ch}\, b^*_{(\alpha_1+\alpha_2,\alpha_1+\alpha_2,\alpha_1)} = [2]_1 11212 + [2]_2[2]_1 11221 + [2]_1 12121,$
$\mathrm{Ch}\, b^*_{(\alpha_2,3\alpha_1+\alpha_2)} = [2]_1[3]_1 21112,$
$\mathrm{Ch}\, b^*_{(\alpha_2,2\alpha_1+\alpha_2,\alpha_1)} = [2]_1 21121,$
$\mathrm{Ch}\, b^*_{(\alpha_2,\alpha_1+\alpha_2,\alpha_1,\alpha_1)} = [2]_1 12121 + [2]_2[2]_1 12211 + [2]_1 21211,$
$\mathrm{Ch}\, b^*_{(\alpha_2,\alpha_2,\alpha_1,\alpha_1,\alpha_1)} = [2]_1[3]_1 21211 + [2]_2[2]_1[3]_1 22111;$
$\mathrm{Ch}\, b^*_{(\alpha_1+\alpha_2)} = 12,$
$\mathrm{Ch}\, b^*_{(\alpha_2,\alpha_1)} = 21;$
$\mathrm{Ch}\, b^*_{(\alpha_2)} = 2.$

Here $[2]_1 = q + q^{-1}$, $[2]_2 = q^3 + q^{-3}$ and $[3]_1 = q^2 + 1 + q^{-2}$.

4.3. **The length two property.** Suppose that $\alpha \in R^+$. The following theorem was stated as a conjecture in [BK, Conjecture 2.16]. It shows that the proper standard module $\bar{\Delta}(\lambda)$ has length two for all minimal pairs λ for α.

Theorem 4.7. *For $\lambda = (\beta, \gamma) \in \mathrm{MP}(\alpha)$ there are short exact sequences*

$$0 \longrightarrow q^{p_{\beta,\gamma} - \beta\cdot\gamma} L(\alpha) \longrightarrow L(\beta) \circ L(\gamma) \longrightarrow L(\lambda) \longrightarrow 0, \tag{4.14}$$

$$0 \longrightarrow q^{-\beta\cdot\gamma} L(\lambda) \longrightarrow L(\gamma) \circ L(\beta) \longrightarrow q^{-p_{\beta,\gamma}} L(\alpha) \longrightarrow 0. \tag{4.15}$$

Proof. Recall the short exact sequence (4.2). The module M has all composition factors isomorphic to $L(\alpha)$ (up to shift). To prove the theorem we need to show that $M \cong q^{-p_{\beta,\gamma}} L(\alpha)$.

Suppose first that α, β, γ lie in a subsystem of type G_2. There are just two convex orderings. For one of these, the character b^*_α of $L(\alpha)$ is listed in Example 4.6, and using this it is easy to check that $\operatorname{Ch} L(\gamma) \circ \operatorname{Ch} L(\beta) = \kappa_\lambda \boldsymbol{i}_\lambda + q^{-p_{\beta,\gamma}} \operatorname{Ch} L(\alpha)$. Comparing with (4.2) we therefore must have that $\operatorname{Dim} 1_{\boldsymbol{i}_\alpha} M + q^{-\beta\cdot\gamma} \operatorname{Dim} 1_{\boldsymbol{i}_\alpha} L(\lambda) = q^{-p_{\beta,\gamma}} \operatorname{Dim} 1_{\boldsymbol{i}_\alpha} L(\alpha)$. The only way this could happen is if $1_{\boldsymbol{i}_\alpha} L(\lambda) = 0$ and $M \cong q^{-p_{\beta,\gamma}} L(\alpha)$. The argument for the other convex ordering is similar.

From now on we assume we are not in G_2. Next we treat the case that $d_\alpha = d_\beta = d_\gamma$, when $p_{\beta,\gamma} = 0$ by (4.3). Applying the functor $\operatorname{Hom}_{H_\alpha}(-, L(\alpha))$ to the short exact sequence (4.2), using Frobenius reciprocity and Corollary 4.3, we deduce that

$$\operatorname{Hom}_{H_\alpha}(M, L(\alpha)) \cong \operatorname{Hom}_{H_\gamma \otimes H_\beta}(L(\gamma) \boxtimes L(\beta), \operatorname{res}^\alpha_{\gamma,\beta} L(\alpha)) \cong \mathbb{K}.$$

Hence M has irreducible head $L(\alpha)$. Therefore by Corollary 3.5 we have that $M \cong \Delta_n(\alpha)$ for some $n \geq 1$, i.e. $[M] = \frac{1-q_\alpha^{2n}}{1-q_\alpha^2}[L(\alpha)]$. To prove the theorem we must show that $n = 1$. For this we compute $r^*_\gamma r^*_\beta - q^{-\beta\cdot\gamma} r^*_\beta r^*_\gamma$ first from (4.1)–(4.2) then from (4.5) to deduce that

$$\frac{1-q_\alpha^{2n}}{1-q_\alpha^2} - q^{-2\beta\cdot\gamma}\frac{1-q_\alpha^{-2n}}{1-q_\alpha^{-2}} = 1 - q^{-2\beta\cdot\gamma}.$$

This easily implies that $n = 1$.

Next we treat the case that $d_\alpha > d_\beta = d_\gamma$, when $p_{\beta,\gamma} = 1$ and $\beta \cdot \gamma = 0$. Then $\left[\operatorname{res}^\alpha_{\gamma,\beta} L(\alpha)\right] = (q + q^{-1})[L(\gamma) \boxtimes L(\beta)]$, and the above calculation and Corollary 3.5 show that $M \cong q^{-1}\Delta_n(\alpha) \oplus q\Delta_m(\alpha)$ for $n \geq 1$ and $m \geq 0$. As above, we obtain the identity

$$q^{-1}\frac{1-q^{4n}}{1-q^4} - q\frac{1-q^{-4n}}{1-q^{-4}} + q\frac{1-q^{4m}}{1-q^4} - q^{-1}\frac{1-q^{-4m}}{1-q^{-4}} = q^{-1} - q,$$

which implies $n = 1$ and $m = 0$ as required.

We are left with the case that β and γ are of different lengths in a subsystem of type B_r, C_r or F_4, when $p_{\beta,\gamma} = 0$ and $\beta \cdot \gamma = -2$. Unfortunately here the above method breaks down: it shows only that $M \cong L(\alpha)$ (as required) or that $M \cong \Delta_2(\alpha)$. To rule out the latter possibility in types B_r or C_r it is enough using (4.2) and Corollary 4.3 to show that $\operatorname{res}^\alpha_{\gamma,\beta} L(\gamma) \circ L(\beta) \cong L(\gamma) \boxtimes L(\beta)$. Realizing R^+ as $\{\epsilon_i \pm \epsilon_j, d\epsilon_k \mid 1 \leq i < j \leq r, 1 \leq k \leq r\}$ in the standard way, where $d = 1$ for B_r or 2 for C_r, the assumptions $\beta + \gamma \in R^+$ and $d_\beta \neq d_\gamma$ imply that $\{\beta, \gamma\} = \{\epsilon_i - \epsilon_j, d\epsilon_j\}$ for some $1 \leq i < j \leq r$. Now apply Theorem 2.1, noting that there is only one non-zero layer in the resulting filtration.

Finally we treat F_4 for β and γ of different lengths. Here we must resort to some explicit calculation. However there are now 2,144,892 different convex orderings! The following argument avoids the need to make a separate computation for each one in turn. Suppose for a contradiction that $M \cong \Delta_2(\alpha)$. Then (4.1) implies that $[L(\lambda)] = r^*_\beta r^*_\gamma - (1 + q^2) r^*_\alpha$. This is b^*-invariant, as is $r^*_\alpha = b^*_{(\alpha)}$, hence $r^*_\beta r^*_\gamma - q^2 r^*_\alpha$ is b^*-invariant too. We deduce from (4.11) that $b^*_\lambda = r^*_\beta r^*_\gamma - q^2 r^*_\alpha$. This shows that $\operatorname{Ch} L(\lambda) = \operatorname{Ch} b^*_\lambda - \operatorname{Ch} b^*_{(\alpha)}$, hence $\operatorname{Ch}(b^*_\lambda - b^*_{(\alpha)})$ is an $\mathbb{N}[q, q^{-1}]$-linear combination of words in $\mathscr{A}\langle I \rangle$. Now we made an explicit computer calculation of the dual canonical basis of $\mathbf{f}^*_\alpha$, using the algorithm in the previous subsection with respect to a particular choice of convex ordering, showing at the end that no two dual canonical basis elements of $\mathbf{f}_\alpha$ have positive difference in this sense; see [B] for details. This produces the desired contradiction. $\square$

Corollary 4.8. *For $\lambda \in \mathrm{MP}(\alpha)$ we have that $[L(\lambda)] = b^*_\lambda$.*

Proof. By (4.14) we have that $[L(\lambda)] = r^*_\beta r^*_\gamma - q^{p_{\beta,\gamma}-\beta\cdot\gamma} r^*_\alpha$, and $p_{\beta,\gamma} - \beta\cdot\gamma > 0$ by Lemma 4.1. Hence using the characterization (4.11) this is also b^*_λ. □

4.4. **A short exact sequence.** In this subsection we fix $\alpha \in R^+$ of height $n \geq 2$. Let (β, γ) be a minimal pair for α and set $m := \mathrm{ht}(\gamma)$.

Lemma 4.9. *Let $w \in S_n$ be $(1, \ldots, n) \mapsto (n - m + 1, \ldots, n, 1, \ldots, n - m)$, so that $\tau_w 1_{\gamma,\beta} = 1_{\beta,\gamma}\tau_w$. There is a unique homogeneous homomorphism*

$$\varphi : q^{-\beta\cdot\gamma}\Delta(\beta) \circ \Delta(\gamma) \to \Delta(\gamma) \circ \Delta(\beta)$$

such that $\varphi(1_{\beta,\gamma} \otimes (v_1 \otimes v_2)) = \tau_w 1_{\gamma,\beta} \otimes (v_2 \otimes v_1)$ for all $v_1 \in \Delta(\beta), v_2 \in \Delta(\gamma)$.

Proof. It suffices by Frobenius reciprocity to show that there is an isomorphism

$$q^{-\beta\cdot\gamma}\Delta(\beta) \boxtimes \Delta(\gamma) \xrightarrow{\sim} \mathrm{res}^\alpha_{\beta,\gamma}\Delta(\gamma) \circ \Delta(\beta), \quad v_1 \otimes v_2 \mapsto \tau_w 1_{\gamma,\beta} \otimes (v_2 \otimes v_1).$$

To see this we apply Theorem 2.1. Suppose we are given $\beta_1, \beta_2, \gamma_1, \gamma_2 \in Q^+$ such that $\gamma = \gamma_1 + \gamma_2 = \gamma_2 + \beta_2, \beta = \beta_1 + \beta_2 = \gamma_1 + \beta_1$, and both of the restrictions $\mathrm{res}^\gamma_{\gamma_1,\gamma_2}\Delta(\gamma)$ and $\mathrm{res}^\beta_{\beta_1,\beta_2}\Delta(\beta)$ are non-zero. By Lemma 2.10, γ_1 is a sum of positive roots $\preceq \gamma \prec \beta$ and β_1 is a sum of positive roots $\preceq \beta$. Since $\gamma_1 + \beta_1 = \beta$ we deduce from Lemma 2.4 that $\beta_1 = \beta, \beta_2 = 0, \gamma_1 = 0$ and $\gamma_2 = \gamma$. Thus the only non-zero layer in the Mackey filtration is the top layer, which is isomorphic to $q^{-\beta\cdot\gamma}\Delta(\beta) \boxtimes \Delta(\gamma)$. □

Theorem 4.10. *For $(\beta, \gamma) \in \mathrm{MP}(\alpha)$ there is a short exact sequence*

$$0 \longrightarrow q^{-\beta\cdot\gamma}\Delta(\beta) \circ \Delta(\gamma) \xrightarrow{\varphi} \Delta(\gamma) \circ \Delta(\beta) \longrightarrow [p_{\beta,\gamma} + 1]\Delta(\alpha) \longrightarrow 0.$$

Proof. In Lemma 4.9, we have already constructed the map φ. We need to show that it is injective and compute its cokernel. Let $V := q^{-\beta\cdot\gamma}\Delta(\beta) \cdot \Delta(\gamma)$ and $W := \Delta(\gamma) \circ \Delta(\beta)$. The endomorphism x of $\Delta(\beta)$ from Lemma 3.9 induces injective endomorphisms

$$y := x \circ 1 \in \mathrm{End}_{H_\alpha}(V)_{2d_\beta}, \qquad y := 1 \circ x \in \mathrm{End}_{H_\alpha}(W)_{2d_\beta}.$$

Similarly the endomorphism x of $\Delta(\gamma)$ gives us

$$z := 1 \circ x \in \mathrm{End}_{H_\alpha}(V)_{2d_\gamma}, \qquad z := x \circ 1 \in \mathrm{End}_{H_\alpha}(W)_{2d_\gamma}.$$

We then have that $y \circ z = z \circ y$, $\varphi \circ y = y \circ \varphi$ and $\varphi \circ z = z \circ \varphi$. Thus we have defined algebra embeddings $\mathbb{K}[y, z] \hookrightarrow \mathrm{End}_{H_\alpha}(V)$ and $\mathbb{K}[y, z] \hookrightarrow \mathrm{End}_{H_\alpha}(W)$.

Let $\mathbb{K}[y, z] = I_0 \supset I_1 \supset \cdots$ be a chain of ideals of $\mathbb{K}[y, z]$ such that for all $m \geq 0$ there exist $b, c \geq 0$ with $I_m = \mathbb{K}y^b z^c \oplus I_{m+1}$, and set $d_m := bd_\beta + cd_\gamma$ for b, c associated to m in this way. This chain of ideals induces filtrations $V = V_0 \supset V_1 \supset \cdots$ and $W = W_0 \supset W_1 \supset \cdots$ with $V_m := I_m(V)$ and $W_m := I_m(W)$. The sections of these filtrations are $V_m/V_{m+1} \cong q^{2d_m-\beta\cdot\gamma}L(\beta) \circ L(\gamma)$ and $W_m/W_{m+1} \cong q^{2d_m}L(\gamma) \circ L(\beta)$. Recalling (4.1)–(4.2), the map φ induces a map $V_m/V_{m+1} \to W_m/W_{m+1}$ which corresponds to the unique (up to a scalar) map $q^{2d_m-\beta\cdot\gamma}L(\beta) \circ L(\gamma) \to q^{2d_m}L(\gamma) \circ L(\beta)$ sending the head of the first onto the socle of the second. Hence φ induces a bijection between occurrences of $L(\lambda)$ as sections of a filtration of V and of W. This shows that the cokernel of φ can only have irreducible subquotients of the form $L(\alpha)$ (up to degree shift).

Now observe using Corollary 4.3 and Theorem 3.3(2) that

$$\begin{aligned}\operatorname{Dim}\operatorname{Hom}_{H_\alpha}(\operatorname{coker}\varphi, L(\alpha)) &\leq \operatorname{Dim}\operatorname{Hom}_{H_\alpha}(\Delta(\gamma)\circ\Delta(\beta), L(\alpha))\\ &= \operatorname{Dim}\operatorname{Hom}_{H_\gamma\otimes H_\beta}(\Delta(\gamma)\boxtimes\Delta(\beta), \operatorname{res}^\alpha_{\gamma,\beta}L(\alpha))\\ &\leq [p_{\beta,\gamma}+1].\end{aligned}$$

Applying Corollary 3.5, we deduce that $\operatorname{Dim}\operatorname{coker}\varphi \leq [p_{\beta,\gamma}+1]\operatorname{Dim}\Delta(\alpha)$. Then by (4.4) we have that

$$\begin{aligned}[p_{\beta,\gamma}+1]\operatorname{Dim}\Delta(\alpha) &= \operatorname{Dim}\Delta(\gamma)\circ\Delta(\beta) - q^{-\beta\cdot\gamma}\operatorname{Dim}\Delta(\beta)\circ\Delta(\gamma)\\ &= \operatorname{Dim}\operatorname{coker}\varphi - \operatorname{Dim}\ker\varphi \leq \operatorname{Dim}\operatorname{coker}\varphi \leq [p_{\beta,\gamma}+1]\operatorname{Dim}\Delta(\alpha).\end{aligned}$$

Equality must hold in both places, showing that $\ker\varphi = 0$ hence φ is injective, and $\operatorname{Dim}\operatorname{coker}\varphi = [p_{\beta,\gamma}+1]\operatorname{Dim}\Delta(\alpha)$. Finally, now that we have worked out both the head of $\operatorname{coker}\varphi$ and its graded dimension, another application of Corollary 3.5 shows indeed that $\operatorname{coker}\varphi \cong [p_{\beta,\gamma}+1]\Delta(\alpha)$. □

Corollary 4.11. *For $\alpha \in Q^+$ of height n and $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathrm{KP}(\alpha)$, the projective dimension of $\Delta(\lambda)$ satisfies $\operatorname{pd}\Delta(\lambda) \leq n-l$.*

Proof. We need to show that $\operatorname{ext}^d_{H_\alpha}(\Delta(\lambda), V) = 0$ for any H_α-module V and $d > n-l$. Using Theorem 3.11 and generalized Frobenis reciprocity, this reduces to checking in the case that α is a positive root that $\operatorname{ext}^d_{H_\alpha}(\Delta(\alpha), V) = 0$ for all $d > n-1$. To see this, apply $\hom_{H_\alpha}(-, V)$ to the short exact sequence from Theorem 4.10 and use generalized Frobenius reciprocity and induction. □

4.5. **Projective resolutions.** Implicit in the proof of Corollary 4.11 are some interesting projective resolutions. To explain this, we again fix a choice of minimal pairs $\mathrm{mp}(\alpha) \in \mathrm{MP}(\alpha)$ for each $\alpha \in R^+$ of height at least two, and define κ_α and κ_λ as in (4.12). Let

$$\tilde\Delta(\alpha) := \kappa_\alpha\Delta(\alpha), \qquad \tilde\Delta(\lambda) := \kappa_\lambda\Delta(\lambda). \tag{4.16}$$

In the next paragraph, we construct a projective resolution $P_*(\alpha)$ of $\tilde\Delta(\alpha)$ for each $\alpha \in R^+$, i.e. a complex

$$\cdots \to P_2(\alpha) \overset{\partial_2}{\to} P_1(\alpha) \overset{\partial_1}{\to} P_0(\alpha) \overset{\partial_0}{\to} 0$$

of projective modules with $\mathrm{H}_0(P_*(\alpha)) \cong \tilde\Delta(\alpha)$ and $\mathrm{H}_d(P_*(\alpha)) = 0$ for $d \neq 0$. More generally, given $\alpha \in Q^+$ of height n and $\lambda = (\lambda_1,\dots,\lambda_l) \in \mathrm{KP}(\alpha)$, the total complex of the tensor product of the complexes $P_*(\lambda_1),\dots,P_*(\lambda_l)$ gives a projective resolution $P_*(\lambda)$ of $\tilde\Delta(\lambda)$. It will be clear from its definition that $P_d(\lambda) = 0$ for $d > n-l$, consistent with Corollary 4.11.

The construction of $P_*(\alpha)$ is recursive. To start with for $i \in I$, we have that $\tilde\Delta(\alpha_i) = H_{\alpha_i}$, which is projective already. So we just have to set $P_0(\alpha_i) := H_{\alpha_i}$ and $P_d(\alpha_i) := 0$ for $d \neq 0$ to obtain the required resolution. Now suppose that $\alpha \in R^+$ is of height at least two and let $(\beta,\gamma) := \mathrm{mp}(\alpha)$. We may assume by induction that the projective resolutions $P_*(\beta)$ and $P_*(\gamma)$ are already defined. Taking the total complex of their tensor product using [W, Acyclic Assembly Lemma 2.7.3], we obtain a projective resolution $P_*(\beta,\gamma)$ of $\tilde\Delta(\beta)\circ\tilde\Delta(\gamma)$ with

$$\begin{aligned}P_d(\beta,\gamma) &:= \bigoplus_{d_1+d_2=d} P_{d_1}(\beta)\circ P_{d_2}(\gamma),\\ \partial_d &:= \left(\operatorname{id}\circ\partial_{d_2} - (-1)^{d_2}\partial_{d_1}\circ\operatorname{id}\right)_{d_1+d_2=d} : P_d(\beta,\gamma) \to P_{d-1}(\beta,\gamma).\end{aligned}$$

Similarly we obtain a projective resolution $P_*(\gamma,\beta)$ of $\tilde{\Delta}(\gamma)\circ\tilde{\Delta}(\beta)$ with

$$P_d(\gamma,\beta) := \bigoplus_{d_1+d_2=d} P_{d_1}(\gamma)\circ P_{d_2}(\beta),$$
$$\partial_d := \left(\partial_{d_1}\circ \mathrm{id} + (-1)^{d_1}\,\mathrm{id}\circ\partial_{d_2}\right)_{d_1+d_2=d} : P_d(\gamma,\beta)\to P_{d-1}(\gamma,\beta).$$

(We've chosen signs carefully here so that Theorem 4.12 works out nicely.) There is an injective homomorphism

$$\tilde{\varphi} : q^{-\beta\cdot\gamma}\tilde{\Delta}(\beta)\circ\tilde{\Delta}(\gamma)\hookrightarrow\tilde{\Delta}(\gamma)\circ\tilde{\Delta}(\beta)$$

defined in exactly the same way as the map φ in Lemma 4.9, indeed, it is just a direct sum of copies of the map φ from there. Applying [W, Comparision Theorem 2.2.6], $\tilde{\varphi}$ lifts to a chain map $\tilde{\varphi}_* : q^{-\beta\cdot\gamma}P_*(\beta,\gamma)\to P_*(\gamma,\beta)$. Then we take the mapping cone of $\tilde{\varphi}_*$ to obtain a complex $P_*(\alpha)$ with

$$P_d(\alpha) := P_d(\gamma,\beta)\oplus q^{-\beta\cdot\gamma}P_{d-1}(\beta,\gamma),$$
$$\partial_d := (\partial_d, \partial_{d-1} + (-1)^{d-1}\tilde{\varphi}_{d-1}) : P_d(\alpha)\to P_{d-1}(\alpha).$$

In view of Theorem 4.10 and [W, Acyclic Assembly Lemma 2.7.3] once again, $P_*(\alpha)$ is a projective resolution of $\tilde{\Delta}(\alpha)$.

Let us describe $P_*(\alpha)$ more explicitly. First for $i\in I$ and the empty tuple σ, set $\boldsymbol{i}_{\alpha_i,\sigma} := i$. Now suppose that α is of height $n\geq 2$ and that $(\beta,\gamma) = \mathrm{mp}(\alpha)$ with γ of height m. For $\sigma = (\sigma_1,\dots,\sigma_{n-1})\in\{0,1\}^{n-1}$, let $|\sigma| := \sigma_1+\cdots+\sigma_{n-1}$, $\sigma_{<m} := (\sigma_1,\dots,\sigma_{m-1})$ and $\sigma_{>m} := (\sigma_{m+1},\dots,\sigma_{n-1})$. Define $\boldsymbol{i}_{\alpha,\sigma}\in\langle I\rangle_\alpha$ and $d_{\alpha,\sigma}\in\mathbb{N}$ recursively from

$$\boldsymbol{i}_{\alpha,\sigma} := \begin{cases} \boldsymbol{i}_{\gamma,\sigma_{<m}}\boldsymbol{i}_{\beta,\sigma_{>m}} & \text{if } \sigma_m = 0,\\ \boldsymbol{i}_{\beta,\sigma_{>m}}\boldsymbol{i}_{\gamma,\sigma_{<m}} & \text{if } \sigma_m = 1;\end{cases}$$
$$d_{\alpha,\sigma} := \begin{cases} d_{\beta,\sigma_{>m}} + d_{\gamma,\sigma_{<m}} & \text{if } \sigma_m = 0,\\ d_{\beta,\sigma_{>m}} + d_{\gamma,\sigma_{<m}} - \beta\cdot\gamma & \text{if } \sigma_m = 1.\end{cases}$$

Note in particular that $d_{\alpha,\sigma} = |\sigma|$ in simply-laced types. Also if $\sigma = (0,\dots,0)$ then $\boldsymbol{i}_{\alpha,\sigma}$ is the tuple $\boldsymbol{i}_\alpha$ from (4.12) and $d_{\alpha,\sigma} = 0$. Then we have that

$$P_d(\alpha) = \bigoplus_{\substack{\sigma\in\{0,1\}^{n-1}\\ |\sigma|=d}} q^{d_{\alpha,\sigma}}H_\alpha 1_{\boldsymbol{i}_{\alpha,\sigma}}. \tag{4.17}$$

For the differentials $\partial_d : P_d(\alpha)\to P_{d-1}(\alpha)$, there are elements $\tau_{\sigma,\rho}\in 1_{\boldsymbol{i}_{\alpha,\sigma}}H_\alpha 1_{\boldsymbol{i}_{\alpha,\rho}}$ for each $\sigma,\rho\in\{0,1\}^{n-1}$ with $|\sigma| = d, |\rho| = d-1$ such that, on viewing elements of (4.17) as row vectors, the differential ∂_d is defined by right multiplication by the matrix $\left(\tau_{\sigma,\rho}\right)_{|\sigma|=d,|\rho|=d-1}$. Moreover $\tau_{\sigma,\rho} = 0$ unless the tuples σ and ρ differ in just one entry. Unfortunately we have not been able to find a satisfactory description of such elements $\tau_{\sigma,\rho}$, except in the following special case.

Theorem 4.12. *Suppose that $\alpha\in R^+$ is multiplicity-free, so that $\kappa_\alpha = 1$ and $P_*(\alpha)$ is a projective resolution of the root module $\Delta(\alpha)$ itself. Then the elements $\tau_{\sigma,\rho}$ inducing the differential $\partial_d : P_d(\alpha)\to P_{d-1}(\alpha)$ as above may be chosen so that*

$$\tau_{\sigma,\rho} := (-1)^{\sigma_1+\cdots+\sigma_{r-1}}\tau_w$$

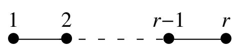
if σ and ρ differ just in the rth entry, where $w\in S_n$ is the unique *permutation with $1_{\boldsymbol{i}_{\alpha,\sigma}}\tau_w = \tau_w 1_{\boldsymbol{i}_{\alpha,\rho}}$.*

Proof. This goes by induction on height. The key point for the induction step is that the chain map $\tilde{\varphi}_* : q^{-\beta\cdot\gamma} P_*(\beta,\gamma) \to P_*(\gamma,\beta)$ in the above construction can be chosen so that $\tilde{\varphi}_d : q^{-\beta\cdot\gamma} P_{d_1}(\beta) \circ P_{d_2}(\gamma) \to P_{d_2}(\gamma) \circ P_{d_1}(\beta)$ is defined by right multiplication by $(-1)^{d_1}\tau_w$, where w is the permutation from Lemma 4.9. The proof that this is indeed a chain map relies on the fact that the braid relations hold exactly in H_α under the assumption that α is multiplicity-free. $\square$

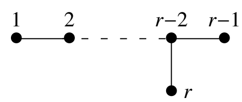
Appendix: Simply-laced types

In this appendix we perform the calculations needed to fill in the gap in the proof of Theorem 3.1 in the simply-laced types. We assume that $\mathfrak{g}$ is of type A_r, D_r or E_r and index the simple roots by $I = \{1, \dots, r\}$ as follows:

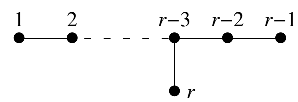
$\mathrm{A}_r \quad (r \geq 1)$



$\mathrm{D}_r \quad (r \geq 4)$



$\mathrm{E}_r \quad (r = 6, 7, 8)$



The natural ordering on I induces a lexicographic total order $<$ on the set $\langle I \rangle$ of words. In [Le, §4.3], Leclerc shows that there is a well-defined injective map $l : R^+ \hookrightarrow \langle I \rangle$ defined recursively by setting $l(\alpha_i) := i$ for each $i \in I$, then

$$l(\alpha) := \max\left(l(\gamma)l(\beta) \mid \beta, \gamma \in R^+, \beta + \gamma = \alpha, l(\beta) > l(\gamma)\right)$$

for non-simple $\alpha \in R^+$. The words $\{l(\alpha) | \alpha \in R^+\}$ are the *good Lyndon words* associated to the order $<$. We define the corresponding *Lyndon ordering* $\prec$ on R^+ by declaring that $\alpha \prec \beta$ if and only if $l(\alpha) < l(\beta)$. This is known to be a convex ordering thanks to a general result from [Ro]. In the following examples we make it explicit by listing the words $\{l(\alpha) \mid \alpha \in R^+\}$ in each type.

Example A.1. In type A_r for the above numbering of the simple roots, the positive roots are $\alpha_{i,j} := \alpha_i + \cdots + \alpha_j$ for $1 \leq i \leq j \leq r$. The good Lyndon word $l(\alpha_{i,j})$ is the *increasing segment* $i \cdots j$. Thus the convex ordering $\prec$ satisfies $\alpha_{i,j} \prec \alpha_{k,l}$ if and only if $i < k$, or $i = k$ and $j < l$. The corresponding reduced expression for w_0 is $(s_1 \cdots s_r)(s_1 \cdots s_{r-1}) \cdots (s_1 s_2) s_1$.

Example A.2. In type D_r the good Lyndon words corresponding to the positive roots are the words $i \cdots j$ for $1 \leq i \leq j \leq r-1$ and the words $i \cdots (r-2)\underline{r \cdots j}$ for $1 \leq i < j \leq r$ (where the underline denotes a *decreasing segment*).

Example A.3. In E_6 the good Lyndon words arranged in lexicographic order are:
1, 12, 123, 1234, 12345, 1236, 12364, 123643, 1236432, 123645, 1236453, 12364532, 12364534, 123645342, 1236453423, 12364534236, 2, 23, 234, 2345, 236, 2364, 23643, 23645, 236453, 2364534, 3, 34, 345, 36, 364, 3645, 4, 45, 5, 6.

Example A.4. In E_7 the good Lyndon words are:
1, 12, 123, 1234, 12345, 123456, 12347, 123475, 1234754, 12347543, 123475432, 1234756, 12347564, 123475643, 1234756432, 123475645, 1234756453, 12347564532, 12347564534, 123475645342, 1234756453423, 123475645347, 1234756453472, 12347564534723, 123475645347234, 1234756453472345, 12347564534723456, 2, 23, 234, 2345, 23456, 2347, 23475, 234754, 2347543, 234756, 2347564, 23475643, 23475645, 234756453, 2347564534, 23475645347, 3, 34, 345, 3456, 347, 3475, 34754, 34756, 347564, 3475645, 4, 45, 456, 47, 475, 4756, 5, 56, 6, 7.

Example A.5. In E_8 the good Lyndon words are:
1, 12, 123, 1234, 12345, 123456, 1234567, 123458, 1234586, 12345865, 123458654, 1234586543, 12345865432, 12345867, 123458675, 1234586754, 12345867543, 123458675432, 1234586756, 12345867564, 123458675643, 1234586756432, 123458675645, 1234586756453, 12345867564532, 12345867564534, 123458675645342, 1234586756453423, 12345867564534231234586756458, 1234586756458, 12345867564583, 123458675645832, 123458675645834, 1234586756458342, 12345867564583423, 1234586756458345, 12345867564583452, 123458675645834523, 1234586756458345234, 12345867564583456, 123458675645834562, 1234586756458345623, 12345867564583456234, 123458675645834562345, 1234586756458345623458, 123458675645834567, 1234586756458345672, 12345867564583456723, 123458675645834567234, 1234586756458345672345, 12345867564583456723456, 12345867564583456723458, 123458675645834567234586, 1234586756458345672345865, 12345867564583456723458654, 123458675645834567234586543, 1234586756458345672345865432, 2, 23, 234, 2345, 23456, 234567, 23458, 234586, 2345865, 23458654, 234586543, 2345867, 23458675, 234586754, 2345867543, 234586756, 2345867564, 23458675643, 23458675645, 234586756453, 2345867564534, 234586756458, 2345867564583, 23458675645834, 234586756458345, 2345867564583456, 23458675645834567, 3, 34, 345, 3456, 34567, 3458, 34586, 345865, 3458654, 345867, 3458675, 34586754, 34586756, 345867564, 3458675645, 34586756458, 4, 45, 456, 4567, 458, 4586, 45865, 45867, 458675, 4586756, 5, 56, 567, 58, 586, 5867, 6, 67, 7, 8.

Working always now with the Lyndon ordering $\prec$ just defined, we also fix the following choice of a minimal pair $\mathrm{mp}(\alpha) \in \mathrm{MP}(\alpha)$ for each $\alpha \in R^+$ of height at least two: let it be the two-part Kostant partition (β, γ) of α for which γ is as big as possible.

Lemma A.6. *Suppose that $\alpha \in R^+$ is of height at least two and let $(\beta, \gamma) := \mathrm{mp}(\alpha)$. Apart from the highest root in* E_8*, the word $l(\gamma)$ is obtained from $l(\alpha)$ by removing its last letter, and $l(\beta)$ is the singleton consisting just of this last letter. For the highest root in* E_8*, we have that $l(\gamma) = 1234586756453423$ and $l(\beta) = 1234586756458$. In all cases, $l(\alpha) = l(\gamma)l(\beta)$ is the costandard factorization of $l(\alpha)$ from* [Le, §3.2]*, and the word $l(\alpha)$ coincides with the word $\boldsymbol{i}_\alpha$ from (4.12).*

Proof. This follows by inspection of the data in the examples. □

Finally we recall the construction of *homogeneous representations* from [KR1]. Let $\sim$ be the equivalence relation on $\langle I \rangle$ generated by interchanging an adjacent pair of letters i and j which are not connected by an edge in the Dynkin diagram. A word $\boldsymbol{i} \in \langle I \rangle$ of length n is said to be *homogeneous* if it is impossible to find $\boldsymbol{j} \sim \boldsymbol{i}$ such that either $j_r = j_{r+1}$ for some $1 \leq r \leq n-1$ or $j_s = j_{s+2}$ for some $1 \leq s \leq n-2$. If $\alpha \in R^+$ is such that $l(\alpha)$ is homogeneous, then the module $L(\alpha)$ can be constructed explicitly as the graded vector space with basis $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \sim l(\alpha)\}$ concentrated in degree zero, such that each $v_{\boldsymbol{i}}$ is in the $\boldsymbol{i}$-word space, all x_j act as zero, and $\tau_k v_{\boldsymbol{i}} := v_{(k\,k+1)(\boldsymbol{i})}$ if i_k and i_{k+1} are not connected by an edge in the Dynkin diagram, $\tau_k v_{\boldsymbol{i}} := 0$ otherwise.

Lemma A.7. *For all $\alpha \in R^+$ except for the highest root in type* E_8*, the word $l(\alpha)$ listed in Examples A.1–A.5 is homogeneous, hence the module $L(\alpha)$ is a homogeneous representation.*

Proof. Again this follows by checking each case in turn. □

The following lemma is a special case of Theorem 4.7, but of course we cannot use that here as the proof of Theorem 4.7 depends on Theorem 3.1.

Lemma A.8. *For $\alpha \in R^+$ of height at least two and $\lambda = (\beta, \gamma) := \mathrm{mp}(\alpha)$, there are non-split short exact sequences*

$$0 \longrightarrow qL(\alpha) \longrightarrow L(\beta) \circ L(\gamma) \longrightarrow L(\lambda) \longrightarrow 0,$$
$$0 \longrightarrow qL(\lambda) \longrightarrow L(\gamma) \circ L(\beta) \longrightarrow L(\alpha) \longrightarrow 0.$$

Proof. We must show that the module M in (4.2) is isomorphic to $L(\alpha)$. This follows if we can show that $\mathrm{Dim}\, 1_{l(\alpha)}(L(\gamma) \circ L(\beta)) = 1$. We know by Lemmas A.6–A.7 that $l(\alpha) = l(\gamma)l(\beta)$ and

$$\mathrm{Ch}\, L(\gamma) = \sum_{\boldsymbol{k} \sim l(\gamma)} \boldsymbol{k}, \qquad \mathrm{Ch}\, L(\beta) = \sum_{\boldsymbol{j} \sim l(\beta)} \boldsymbol{j}.$$

Hence we are reduced to showing that the only pair $(\boldsymbol{k}, \boldsymbol{j})$ with $\boldsymbol{k} \sim l(\gamma)$ and $\boldsymbol{j} \sim l(\beta)$ such that $l(\alpha)$ has non-zero coefficient in the shuffle product $\boldsymbol{k} \circ \boldsymbol{j}$ is $(\boldsymbol{k}, \boldsymbol{j}) = (l(\gamma), l(\beta))$, and moreover for this pair the only shuffle of $\boldsymbol{k}$ and $\boldsymbol{j}$ that produces $l(\alpha)$ is the identity. Apart from the highest root of E_8, this follows because in all cases no $\boldsymbol{k} \sim l(\gamma)$ ends in the letter $l(\beta)$. The highest root of E_8 takes only a little more combinatorial analysis using the explicit descriptions of $l(\gamma)$ and $l(\beta)$ from Lemma A.6. □

Theorem A.9. *Assume that $\mathfrak{g}$ is simply-laced and that $\prec$ is the Lyndon ordering fixed above. Let $\alpha \in R^+$ have height at least two and set $(\beta, \gamma) := \mathrm{mp}(\alpha)$. Then there exists an H_α-module X such that $X/\mathrm{soc}\, X \cong L(\gamma) \circ L(\beta)$ and $\mathrm{soc}\, X \cong q^2 L(\alpha)$.*

Proof. By Lemma A.7, the modules $L(\beta)$ and $L(\gamma)$ can be constructed explicitly as above as they are homogeneous representations. In a similar way we construct a module $\Delta_2(\beta)$ with $\Delta_2(\beta)/\mathrm{soc}\, \Delta_2(\beta) \cong L(\beta)$ and $\mathrm{soc}\, \Delta_2(\beta) \cong q^2 L(\beta)$, by declaring that it has homogeneous basis $\{v_{\boldsymbol{i}}^{\pm} | \boldsymbol{i} \sim l(\beta)\}$ with $v_{\boldsymbol{i}}^{\pm} \in 1_{\boldsymbol{i}} \Delta_2(\beta)_{1 \pm 1}$, such that $x_j v_{\boldsymbol{i}}^- := v_{\boldsymbol{i}}^+$, $x_j v_{\boldsymbol{i}}^+ := 0$, and $\tau_k v_{\boldsymbol{i}}^{\pm} := v_{(k\, k+1)(\boldsymbol{i})}^{\pm}$ if i_k and i_{k+1} are not connected by an edge in the Dynkin diagram, $\tau_k v_{\boldsymbol{i}}^{\pm} := 0$ otherwise.

Now consider the second short exact sequence from Lemma A.8. Combined with exactness of induction, it follows that $L(\gamma) \circ \Delta_2(\beta)$ has a unique submodule $S \cong q^3 L(\lambda)$. Set $X := L(\gamma) \circ \Delta_2(\beta)/S$ and $v^{\pm} := 1_{\gamma,\beta} \otimes (v_{l(\gamma)} \otimes v_{l(\beta)}^{\pm}) + S$. Then X has a unique submodule $Y \cong q^2 L(\alpha)$ generated by v^+, and $X/Y \cong L(\gamma) \circ L(\beta)$. It remains to show that $Y = \mathrm{soc}\, X$. Suppose for a contradiction that the socle is larger. Then X must also have a submodule $Z \cong qL(\lambda)$. In the next two paragraphs, we prove that there exists a word $\boldsymbol{i} \in \langle I \rangle_\alpha$ and elements $a, b \in H_\alpha$ such that $1_{\boldsymbol{i}} L(\alpha) = 0$, $bv^- \in 1_{\boldsymbol{i}} X$, and $abv^- = v^+$. This is enough to complete the proof, for then we must have that $bv^- \in Z$, hence $v^+ = abv^- \in Z$ too, contradicting $Y \cap Z = 0$.

To construct a, b and $\boldsymbol{i}$, we first assume that α is *not* the highest root in type E_8. Suppose that α is of height n. Let $1 \leq p < n$ be maximal such that the pth letter i of $l(\alpha)$ is connected to its nth letter j in the Dynkin diagram. By inspection of the information in Examples A.1–A.5, this is always possible and moreover none of the letters in between i and j are equal to j. Let w be the cycle $(p\ p+1\ \cdots\ n)$ and set $a := \tau_{w^{-1}}, b := \tau_w$. Finally let $\boldsymbol{i}$ be the word obtained from $l(\alpha)$ by deleting the nth letter j then reinserting it just before the pth letter i; then we have that $bv^- \in 1_{\boldsymbol{i}} X$. An easy application of the relations shows that $abv^- = v^+$ (up to a sign). We are left with showing that $1_{\boldsymbol{i}} L(\alpha) = 0$. But in all these cases $l(\alpha)$ is also homogeneous so this follows as $\boldsymbol{i} \nsim l(\alpha)$ by construction.

It remains to treat the highest root for E_8. Here Lemma A.7 tells us that $l(\beta) = 1234586756458$ and $l(\gamma) = 1234586756453423$, but the word $l(\alpha) = l(\gamma)l(\beta)$ is no longer homogeneous. We set $\boldsymbol{i} := 1234586756453421234586756435 8$, $a := \tau_w$ and $b := \tau_{w^{-1}}$ where w is the cycle $(16\ 17\ \cdots\ 27)$; again we have that $bv^- \in 1_{\boldsymbol{i}}V$. Another explicit relation check (best made now by drawing a picture) shows that $abv^- = v^+$ (up to a sign). It remains to show that $1_{\boldsymbol{i}}L(\alpha) = 0$. From Lemma A.8, we deduce that $(1-q^2)[L(\alpha)] = [L(\gamma) \circ L(\beta)] - q[L(\beta) \circ L(\gamma)]$, hence

$$\operatorname{Ch} L(\alpha) = \sum_{\boldsymbol{k} \sim l(\gamma), \boldsymbol{j} \sim l(\beta)} (\boldsymbol{k} \circ \boldsymbol{j} - q\boldsymbol{j} \circ \boldsymbol{k})/(1-q^2).$$

Now one more calculation shows that the $\boldsymbol{i}$-coefficient on the right hand side is indeed zero. □

Corollary A.10. *Theorem 3.1 holds for simply-laced $\mathfrak{g}$ and the Lyndon ordering $\prec$ defined above.*

Proof. This follows by mimicking the argument explained in the first half of the proof of [M, Proposition 4.5]. The essential ingredients needed for this are provided by Lemma A.8 and Theorem A.9. □

Corollary A.11. *For simply-laced $\mathfrak{g}$ and $\alpha \in Q^+$ of height n, the KLR algebra H_α has finite global dimension n, i.e.* $\sup \operatorname{pd} V = n$ *where the supremum is taken over all H_α-modules V.*

Proof. We choose the convex ordering to be the Lyndon ordering as in Corollary A.10, so that Theorem 3.1 is proved. This is all that is needed for all the subsequent results from sections 3 and 4 to be proved for this ordering. Then we argue as in the proof of [M, Theorem 4.6] to reduce to showing that $\operatorname{ext}^d_{H_\alpha}(L(\lambda), V) = 0$ for any H_α-module V and $d > n$. Since $L(\lambda)$ is the socle of $\bar{\nabla}(\lambda)$ and all its other composition factors are of the form $L(\mu)$ (up to degree shift) for $\mu \prec \lambda$, this follows by induction on the ordering if we can show that $\operatorname{ext}^d_{H_\alpha}(\bar{\nabla}(\lambda), V) = 0$ and $d > n$. To prove this, note by Lemma 2.3 and up to a degree shift that $\bar{\nabla}(\lambda)$ is induced from $L(\lambda_l) \boxtimes \cdots \boxtimes L(\lambda_1)$, so we can use generalized Frobenius reciprocity to reduce further to showing for a positive root α of height n that $\operatorname{ext}^d_{H_\alpha}(L(\alpha), V) = 0$ for any V and $d > n$. Finally this follows by applying $\hom_{H_\alpha}(-, V)$ to (3.3) and using Corollary 4.11. □

Remark A.12. As noted already in the introduction, the global dimension of H_α is equal to $\operatorname{ht}(\alpha)$ in non-simply-laced types too; see [M, Theorem 4.6].

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