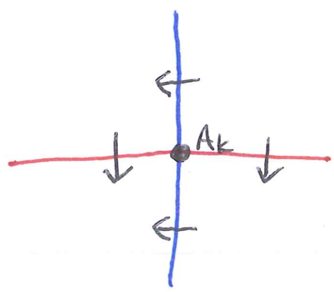
We have the function $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(P) = \sum_{k=1}^{2017} k \frac{\overrightarrow{PA_k}}{\|\overrightarrow{PA_k}\|^5}.$$

Write $F(x,y) = (f(x,y), g(x,y))$.

We will draw the [redacted] zero set of f in blue and the zero set of g in ~~g~~ red.

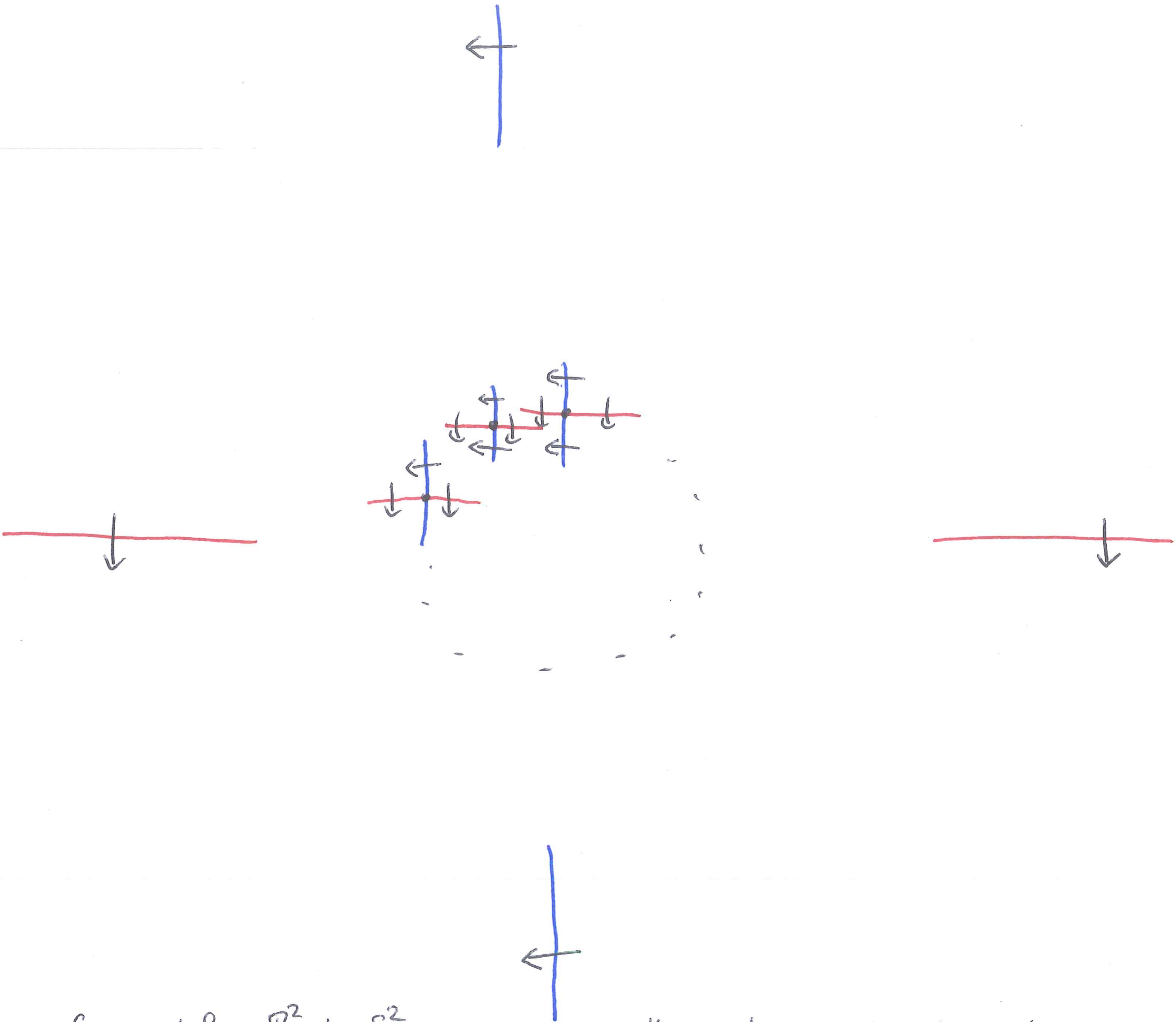
First consider the behaviour of F near a point A_k. In a neighbourhood of A_k, the term $k \cdot \frac{\overrightarrow{PA_k}}{\|PA_k\|^5}$ dominates, so $F(P)$ has essentially the same direction ~~at~~ as $\overrightarrow{PA_k}$. Thus in a neighbourhood of A_k, our drawing looks like



The extra black arrows in the diagram have the following meaning: Crossing a blue (respectively red) line in the direction of the arrow means we are passing from a region where $f(x,y) < 0$ (respectively $g(x,y) < 0$) to one where $f(x,y) > 0$ (respectively $g(x,y) > 0$).

Now consider the behaviour of F as P tends to infinity. In this case, all the vectors $\overrightarrow{PA_k}$ point in essentially the same direction, so the direction of $f(P)$ ~~tends~~ converges to that of $\overrightarrow{PO}$ where $O = (0,0)$. So near infinity our blue curve has two components tending ~~to~~ a vertical direction and our red curve has two horizontal components.

Our picture is now (see next page)



Compactify $\mathbb{R}^2$ to S^2 and consider this picture on the two-sphere. We now claim that the zero set of f is homeomorphic to a finite graph where every vertex has even degree.

Since the zero set of f has been determined outside of a compact set, a standard compactness argument shows ~~we~~ it suffices to prove this statement in the neighbourhood of any point $Q \neq A_k$.

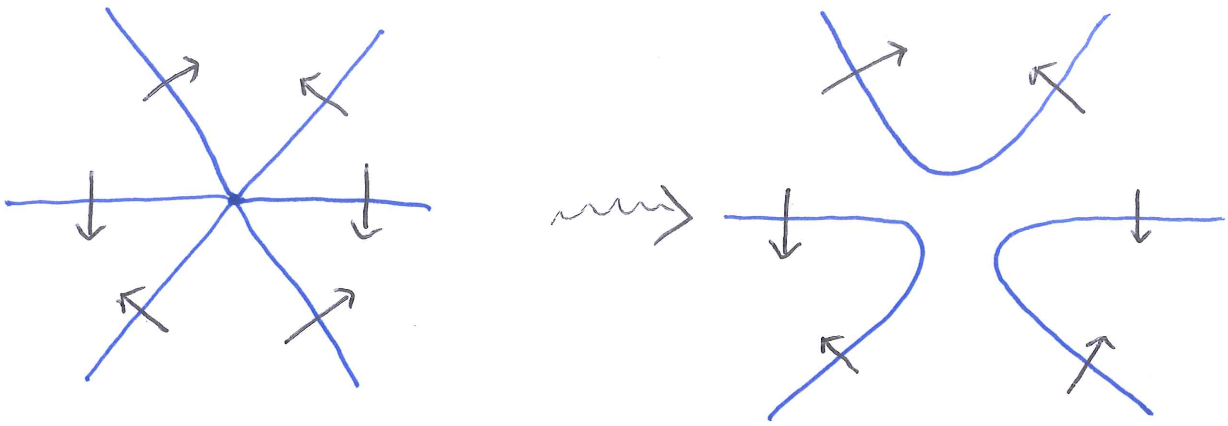
If f is a submersion at Q this claim is obvious. Otherwise since f is real analytic, we can perform a finite number of blowups to reduce to the case where f is a submersion (one should locally factor f first).

The ~~f~~ zero set of g also satisfies the same property.

We now perform some modifications to our graph.

First, remove any edges along which f (or g) vanishes to even order. This allows us to put our black orientation ~~edge~~ arrows along each edge.

Next, modify any edge of degree larger than two in the following way:



We have to show that the (original) red and blue graphs intersect (outside ∞ and $A_1, A_2, \ldots, A_{2017}$).

Assume for want of a contradiction that they do not intersect. Then by ensuring our modifications are small enough, we can assume that our modified graphs do not intersect.

Taken together, our blue and red graphs then define a graph on S^2 with 2018 vertices $(\infty, A_1, A_2, \ldots, A_{2017})$.

Each vertex has degree 4, so there are $\frac{4 \times 2018}{2} = 4036$ edges.

By Euler's formula $V - E + F = 2$, there are 2020 faces.

Let f_k be the number of faces with k edges.

Since edges alternate in colour around a face, $f_k = 0$ if k is odd.

Counting edge-face pairs gives

$$2f_2 + 4f_4 + 6f_6 + 8f_8 + \dots = 2 \times 4036 = 8072.$$

Also we have

$$f_2 + f_4 + f_6 + f_8 + \dots = 2020.$$

$$\therefore\ 2f_2 + 8072 = 4f_2 + 4f_4 + 6f_6 + 8f_8 + \dots$$

$$\geq 4(f_2 + f_4 + f_6 + f_8 + \dots)$$

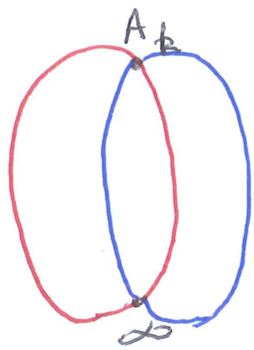
$$= 4 \times 2020 = 8080$$

$$\therefore\ f_2 \geq 4.$$

A consideration of all possible orientations of the black arrows shows it is impossible to have a 2-gon between points A_i and A_j.

$\therefore$ All 2-gons involve the point ∞.

The only way to have four 2-gons with the point ∞ is in the following configuration:



But then these four 2-gons would cover the entire sphere, a contradiction, completing the proof.